

On the Energy and Laplacian Energy of the Shadow, Thorn, and Middle Graphs of the Lintang Graph

Fransiskus Fran¹, Meliana Pasaribu¹, Raventino¹ and Tria Mutiara¹

Abstract—Structural graph transformations can substantially alter spectral quantities even when they are applied to the same parent graph. This study examines this phenomenon for the lintang graph $L_n \cong K_{2,n}$ through three operations: the shadow graph $D_2(L_n)$, the one-thorn graph $T(L_n)$, and the middle graph $M(L_n)$. Their adjacency and Laplacian spectra are derived using complete bipartite graph properties, quadratic spectral relations, structured block matrices, the Schur complement, incidence-matrix relations, and invariant-subspace decompositions. The resulting energies are $E(D_2(L_n)) = 4\sqrt{2n}$, $E(T(L_n)) = 2\sqrt{2n+4} + 2n$, and $E(M(L_n)) = 2(n-1) + 2\sqrt{2}(n-1) + \sqrt{n^2+4} + \sqrt{n^2+4n+8}$. The corresponding Laplacian energies are $LE(D_2(L_n)) = \frac{8(2n^2-3n+2)}{n+2}$, $LE(T(L_n)) = \frac{4n}{n+2} + 2\sqrt{2}(n-1) + \sqrt{n^2+4} + \sqrt{n^2+4n+8}$, and $LE(M(L_n)) = \frac{(n+4)(n^2+3n-2)}{3n+2} + (n-1)\sqrt{n^2+8} + \sqrt{n^2-4n+16} + \sqrt{n^2-4n+8}$. Spectral integrality is also examined because it distinguishes graph transformations that preserve integer-valued spectra from those that necessarily introduce irrational eigenvalues, a distinction of interest in algebraic and chemical graph theory. The shadow graph is Laplacian integral for every $n \geq 2$ and is integral precisely when $n = 2k^2$ for some positive integer k . In contrast, $T(L_n)$ and $M(L_n)$ are neither integral nor Laplacian integral for any $n \geq 2$. These results provide a unified comparison of how neighborhood duplication, pendant extension, and incidence transformation affect the energetic and arithmetic spectral structure of the same parent graph.

I. INTRODUCTION

Structural transformations do more than enlarge a graph; they modify its adjacency pattern, degree distribution, and incidence relations, and these changes are reflected quantitatively in its spectrum. A modification of the adjacency matrix may shift, create, or repeat adjacency eigenvalues, while changes in vertex degrees alter the Laplacian matrix and its eigenvalue distribution. Consequently, graph transformations may change not only the adjacency and Laplacian spectra, but also graph energy, Laplacian energy, and spectral integrality [1], [2]. Graph energy, introduced by Gutman [3], measures the sum of the absolute values of the adjacency eigenvalues, whereas Laplacian energy, introduced by Gutman and Zhou [4], measures the deviation of the Laplacian eigenvalues from the average degree. These invariants therefore provide complementary quantitative descriptions of how a structural operation redistributes spectral information.

*This work received no external funding

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Manuscript received July 27, 2026; accepted September 23, 2026.

Existing studies have approached these topics from several directions. Energy and Laplacian energy have been investigated for bipartite and other graph classes [4], [5], [6], [7], while spectral integrality has been studied as an arithmetic property of graph spectra [8]. For particular graph transformations, shadow graphs have been examined in relation to spectral-type indices and energies [9], thorn graphs have been considered through topological indices and pendant extensions [10], and middle graphs have mainly been studied from structural and domination perspectives [11], [12]. These contributions provide useful results for the individual constructions, but they address different spectral quantities and are developed in separate settings. Consequently, they do not provide a common framework for comparing the adjacency spectrum, Laplacian spectrum, energy, Laplacian energy, and integrality of the shadow, one-thorn, and middle transformations when all three are applied to the same parent graph. This unresolved comparison provides the main motivation for the present study.

The lintang graph is particularly suitable for such a comparison. Throughout this paper, $L_n \cong K_{2,n}$, $n \geq 2$, in accordance with its established structure [13]. It is a nonregular complete bipartite graph with two vertices of degree n and n vertices of degree 2. The choice of $K_{2,n}$ is deliberate: its bipartite structure yields an explicitly tractable adjacency and Laplacian spectrum, while its two distinct vertex degrees are sufficient to produce nontrivial spectral responses after transformation. Thus, L_n provides a controlled base graph on which the influence of different structural operations can be compared without introducing additional complexity from the parent graph.

The three transformations considered here were selected because they modify L_n through fundamentally different mechanisms. The shadow graph $D_2(G)$ duplicates the neighborhood relations across two copies of G [14]. The one-thorn graph $T(G)$ attaches exactly one pendant vertex to every vertex of G [10], thereby representing a local vertex extension while preserving the original graph as an induced subgraph. The middle graph $M(G)$ converts incidence information between vertices and edges into adjacency relations among newly introduced edge-vertices [11], [12]. Hence, the three constructions represent neighborhood duplication, pendant extension, and incidence transformation, respectively. Studying them on the same parent graph makes it possible to distinguish the spectral effect of the transformation from the spectral features already present in L_n .

This distinction is relevant beyond the formal computation of eigenvalues. Spectral descriptors are widely used to encode structural information in graph-based models, including

molecular and network representations [2]. In this setting, energy-related quantities summarize spectral redistribution, while integrality provides an arithmetic criterion for distinguishing transformations whose spectra remain integer-valued from those that necessarily introduce irrational eigenvalues. The comparison therefore clarifies how different graph operations preserve, amplify, or destroy spectral regularity.

Different techniques are required because the three transformed graphs have different algebraic structures. The identity $D_2(L_n) \cong K_{4,2n}$ allows the shadow case to be treated through the standard spectra of complete bipartite graphs. For $T(L_n)$, the natural partition into original and pendant vertices produces block matrices that reduce the spectral problem to quadratic relations involving the eigenvalues of L_n . The middle graph $M(L_n)$ has a more involved incidence structure, so its adjacency spectrum is obtained through a structured block representation and the Schur complement, while its Laplacian spectrum is determined from invariant subspaces induced by the natural vertex classes. These methods are therefore not interchangeable devices; each one reflects the algebraic structure created by the corresponding transformation.

Unlike previous studies that focus on individual transformations or different spectral invariants, this work provides a unified spectral comparison of three structurally distinct transformations on the same base graph. Exact expressions are derived for the energy and Laplacian energy of $D_2(L_n)$, $T(L_n)$, and $M(L_n)$, together with their adjacency and Laplacian integrality properties. The theoretical value of this comparison lies in identifying which spectral features are preserved, amplified, or lost under neighborhood duplication, pendant extension, and incidence transformation. In particular, the analysis shows that the shadow graph is Laplacian integral for every $n \geq 2$ and admits an infinite integral subfamily determined by $n = 2k^2$, whereas the one-thorn and middle constructions necessarily introduce irrational adjacency and Laplacian eigenvalues throughout the admissible range.

II. PRELIMINARIES

This section recalls the graph constructions and spectral notation used throughout the paper. All graphs considered are finite, simple, and undirected.

Definition 1. [13] *The lintang graph L_n , for $n \geq 2$, is the complete bipartite graph $L_n \cong K_{2,n}$.*

Let $U = \{u_1, u_2\}$ and $V = \{v_1, v_2, \dots, v_n\}$ be the two bipartition classes of L_n . Hence, $|V(L_n)| = n + 2$ and $|E(L_n)| = 2n$. The two vertices in U have degree n , whereas each vertex in V has degree 2. For example, $L_3 \cong K_{2,3}$ has $|V(L_3)| = 5$ and $|E(L_3)| = 6$.

Definition 2. [14] *Let G be a graph. The shadow graph $D_2(G)$ is obtained from two copies of G by joining every vertex in each copy to the neighbors of its corresponding vertex in the other copy.*

For $L_n \cong K_{2,n}$, the shadow construction gives $D_2(L_n) \cong K_{4,2n}$. Therefore, $|V(D_2(L_n))| = 2n + 4$ and $|E(D_2(L_n))| = 8n$. In particular, $|V(D_2(L_3))| = 10$ and $|E(D_2(L_3))| = 24$.

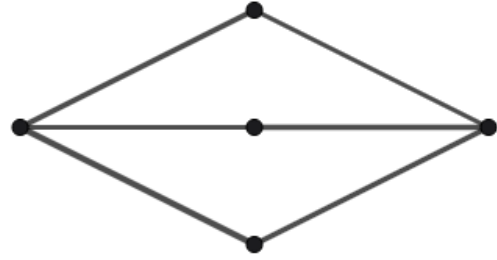


Fig. 1. The lintang graph $L_3 \cong K_{2,3}$.

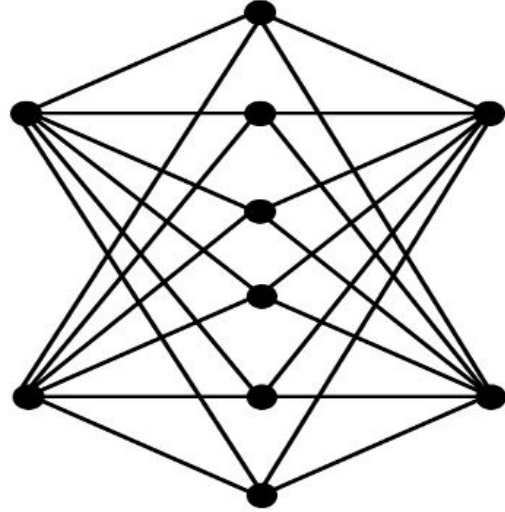


Fig. 2. The shadow graph $D_2(L_3)$.

Definition 3. [10] *Let G be a graph. The one-thorn graph $T(G)$ is obtained by attaching exactly one pendant vertex to each vertex of G .*

Since one new pendant vertex is attached to every vertex of L_n , $|V(T(L_n))| = 2|V(L_n)| = 2n + 4$. Moreover, exactly $n + 2$ new pendant edges are added to the original $2n$ edges, so $|E(T(L_n))| = 2n + (n + 2) = 3n + 2$.

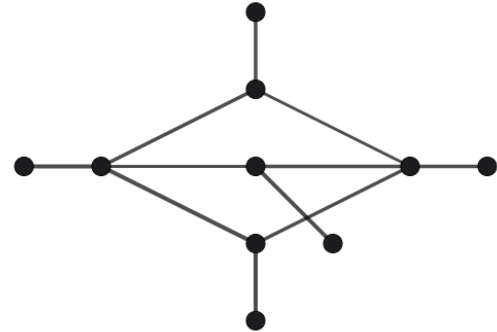


Fig. 3. The one-thorn graph $T(L_3)$.

Definition 4. [11], [12] *Let G be a graph. The middle graph $M(G)$ is obtained by inserting a new vertex into every edge of G and joining two new vertices whenever their corresponding edges in G are incident.*

If G has order p , size q , and vertex degrees $d_1, d_2, \dots, d_p$, then $|V(M(G))| = p + q$. Its edges consist of the $2q$ incidence edges between original vertices and edge-vertices together with the edges joining pairs of incident edges of G . Hence,

$$|E(M(G))| = 2q + \sum_{i=1}^p \binom{d_i}{2}.$$

For L_n , the degree sequence consists of two vertices of degree n and n vertices of degree 2. Therefore, $|V(M(L_n))| = (n + 2) + 2n = 3n + 2$, and

$$\begin{aligned} |E(M(L_n))| &= 4n + 2 \binom{n}{2} + n \binom{2}{2} \\ &= 4n + n(n - 1) + n \\ &= n^2 + 4n. \end{aligned}$$

For $n = 3$, $|V(M(L_3))| = 11$ and $|E(M(L_3))| = 21$.

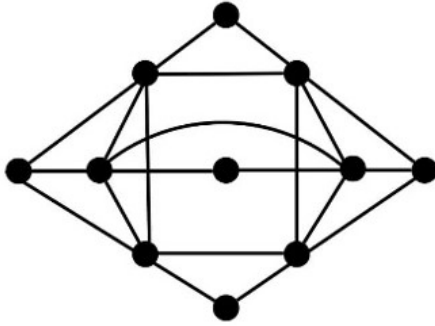


Fig. 4. The middle graph $M(L_3)$.

Definition 5. [15] Let G be a graph of order p . If the distinct adjacency eigenvalues of G are $\lambda_1, \lambda_2, \dots, \lambda_r$ with multiplicities $m_1, m_2, \dots, m_r$, then

$$\text{Spec}_A(G) = \left\{ \lambda_1^{[m_1]}, \lambda_2^{[m_2]}, \dots, \lambda_r^{[m_r]} \right\},$$

where

$$\sum_{i=1}^r m_i = p.$$

Similarly, if the distinct Laplacian eigenvalues are $\mu_1, \mu_2, \dots, \mu_s$ with multiplicities $t_1, t_2, \dots, t_s$, then

$$\text{Spec}_L(G) = \left\{ \mu_1^{[t_1]}, \mu_2^{[t_2]}, \dots, \mu_s^{[t_s]} \right\},$$

where

$$\sum_{i=1}^s t_i = p.$$

These equalities indicate that the multiplicities account for all p adjacency and Laplacian eigenvalues, respectively, when repeated eigenvalues are counted according to their multiplicities.

Definition 6. [3] Let G be a graph of order p with adjacency eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_p$. The energy of G is

$$E(G) = \sum_{i=1}^p |\lambda_i|.$$

Definition 7. [4] Let G be a graph of order p and size q , with Laplacian eigenvalues $\mu_1, \mu_2, \dots, \mu_p$. The Laplacian energy of G is

$$LE(G) = \sum_{i=1}^p |\mu_i - \bar{d}(G)|,$$

where

$$\bar{d}(G) = \frac{2q}{p}$$

is the average degree of G .

Definition 8. [8] A graph G is called integral if every adjacency eigenvalue of G is an integer. It is called Laplacian integral if every Laplacian eigenvalue of G is an integer.

The standard spectra of complete bipartite graphs will be used in the analysis of the shadow graph. For $r, s \geq 1$,

$$\text{Spec}_A(K_{r,s}) = \left\{ \sqrt{rs}, -\sqrt{rs}, 0^{[r+s-2]} \right\},$$

and

$$\text{Spec}_L(K_{r,s}) = \left\{ 0, r^{[s-1]}, s^{[r-1]}, r+s \right\}$$

[15]. In particular, these formulas will be applied to $D_2(L_n) \cong K_{4,2n}$.

Since $L_n \cong K_{2,n}$, the corresponding spectra are

$$\text{Spec}_A(L_n) = \left\{ \sqrt{2n}, -\sqrt{2n}, 0^{[n]} \right\}$$

and

$$\text{Spec}_L(L_n) = \left\{ 0, 2^{[n-1]}, n, n+2 \right\}.$$

III. METHODS

The three graph transformations are analyzed within a common spectral framework, while the algebraic treatment is adapted to the structure created by each operation. The shadow graph is handled through its complete bipartite representation, whereas the one-thorn and middle graphs require structured matrix decompositions.

Research Algorithm. The analytical procedure is carried out as follows.

- 1) Construct $D_2(L_n)$, $T(L_n)$, and $M(L_n)$ from the common parent graph $L_n \cong K_{2,n}$.
- 2) Determine the order, size, degree structure, and average degree of each transformed graph.
- 3) Form suitable adjacency and Laplacian matrix representations according to the structural partition induced by each transformation.
- 4) Determine $\text{Spec}_A(G)$ and $\text{Spec}_L(G)$, and use these spectra to derive $E(G)$ and $LE(G)$.
- 5) Examine the resulting eigenvalues to characterize adjacency and Laplacian integrality.
- 6) Compare the three graph families analytically and use selected numerical values only to illustrate the closed-form results.

For the shadow graph, the structural relation $D_2(L_n) \cong K_{4,2n}$ reduces the spectral analysis to the standard adjacency and Laplacian spectra of complete bipartite graphs recalled in Section II. These spectra are used directly to obtain

$E(D_2(L_n))$, $LE(D_2(L_n))$, and the corresponding integrality properties.

For the one-thorn graph $T(L_n)$, the original vertices of L_n are listed first, followed by their pendant neighbors. Under this ordering, the adjacency matrix is

$$A(T(L_n)) = \begin{pmatrix} A(L_n) & I_{n+2} \\ I_{n+2} & O_{n+2} \end{pmatrix},$$

and the Laplacian matrix is

$$L(T(L_n)) = \begin{pmatrix} L(L_n) + I_{n+2} & -I_{n+2} \\ -I_{n+2} & I_{n+2} \end{pmatrix}.$$

Applying the Schur complement [16] to the corresponding characteristic matrices reduces the spectral problems to quadratic relations. If λ_i is an adjacency eigenvalue of L_n , then the associated adjacency eigenvalues λ of $T(L_n)$ satisfy $\lambda^2 - \lambda_i \lambda - 1 = 0$. Similarly, if θ_i is a Laplacian eigenvalue of L_n , then the corresponding Laplacian eigenvalues μ satisfy $\mu^2 - (\theta_i + 2)\mu + \theta_i = 0$. These relations determine $\text{Spec}_A(T(L_n))$ and $\text{Spec}_L(T(L_n))$, from which $E(T(L_n))$, $LE(T(L_n))$, and the integrality properties are obtained.

For the middle graph, let $\{u_1, u_2\}$ and $\{v_1, v_2, \dots, v_n\}$ be the two bipartition classes of $L_n \cong K_{2,n}$. Order the edges as $u_1v_1, \dots, u_1v_n, u_2v_1, \dots, u_2v_n$, and let B denote the vertex-edge incidence matrix of L_n . With this ordering,

$$B = \begin{pmatrix} \mathbf{1}_n^T & \mathbf{0}_n^T \\ \mathbf{0}_n^T & \mathbf{1}_n^T \\ I_n & I_n \end{pmatrix},$$

where $\mathbf{1}_n$ and $\mathbf{0}_n$ are the all-ones and zero column vectors of length n , respectively.

Using the partition of $V(M(L_n))$ into original vertices and edge-vertices, the adjacency matrix is

$$A(M(L_n)) = \begin{pmatrix} O_{n+2} & B \\ B^T & B^T B - 2I_{2n} \end{pmatrix}.$$

The degree matrix associated with the original vertices of L_n is $D(L_n) = \text{diag}(n, n, 2I_n)$. Since every edge of L_n joins a vertex of degree n to a vertex of degree 2, each corresponding edge-vertex of $M(L_n)$ has degree $n+2$. Hence,

$$L(M(L_n)) = \begin{pmatrix} D(L_n) & -B \\ -B^T & (n+4)I_{2n} - B^T B \end{pmatrix}.$$

For $M(L_n)$, the adjacency characteristic determinant is reduced through the Schur complement together with the relation between the incidence matrix and the signless Laplacian matrix. The Laplacian spectrum is obtained from invariant subspaces induced by the natural vertex classes of the middle graph. In particular, the decomposition separates the zero-sum component from the class-constant component, leading to the characteristic factors used in Section IV.

The resulting spectra are substituted into Definitions 6 and 7. Integrality is determined from the arithmetic form of the resulting eigenvalues. Finally, the closed-form expressions for $D_2(L_n)$, $T(L_n)$, and $M(L_n)$ are compared analytically, while numerical values for selected n are used only as illustrations and not as the basis for statements valid over the entire range $n \geq 2$.

IV. RESULTS AND DISCUSSION

This section establishes the energy, Laplacian energy, and integrality properties of the three transformed lintang graphs. Throughout the section, $n \geq 2$, in accordance with the parameter range in Definition 1. Thus, although $K_{2,1}$ is a well-defined complete bipartite graph, it does not belong to the lintang family considered in this study [13].

A. The Shadow Graph of the Lintang Graph

Theorem 1. *Let $n \geq 2$. The energy and Laplacian energy of the shadow graph $D_2(L_n)$ are $E(D_2(L_n)) = 4\sqrt{2n}$ and $LE(D_2(L_n)) = \frac{8(2n^2-3n+2)}{n+2}$, respectively. Furthermore, $D_2(L_n)$ is Laplacian integral for every $n \geq 2$, while it is integral precisely when $n = 2k^2$ for some positive integer k .*

Proof. Let $U = \{u_1, u_2\}$, $V = \{v_1, \dots, v_n\}$ be the bipartition classes of $L_n \cong K_{2,n}$, and let $U' = \{u'_1, u'_2\}$, $V' = \{v'_1, \dots, v'_n\}$ be the corresponding classes in a second copy of L_n . Under the shadow construction, every vertex in $U \cup U'$ is adjacent to every vertex in $V \cup V'$, whereas no two vertices within the same combined class are adjacent. Hence, $D_2(L_n) = K_{U \cup U', V \cup V'} \cong K_{4,2n}$.

Using the standard adjacency spectrum of a complete bipartite graph recalled in Section II [15], we obtain

$$\text{Spec}_A(D_2(L_n)) = \{2\sqrt{2n}, -2\sqrt{2n}, 0^{[2n+2]}\}.$$

Therefore, $E(D_2(L_n)) = |2\sqrt{2n}| + |-2\sqrt{2n}| = 4\sqrt{2n}$. Similarly,

$$\text{Spec}_L(D_2(L_n)) = \{0, 4^{[2n-1]}, (2n)^{[3]}, 2n+4\}.$$

Since every eigenvalue in this Laplacian spectrum is an integer for $n \geq 2$, the graph $D_2(L_n)$ is Laplacian integral throughout the admissible range. To determine its Laplacian energy, note that $|V(D_2(L_n))| = 2n+4$ and $|E(D_2(L_n))| = 8n$. Therefore, its average degree is

$$\begin{aligned} \bar{d}(D_2(L_n)) &= \frac{2|E(D_2(L_n))|}{|V(D_2(L_n))|} \\ &= \frac{2(8n)}{2n+4} \\ &= \frac{8n}{n+2}. \end{aligned}$$

For brevity, write $\bar{d} = \bar{d}(D_2(L_n))$. For every $n \geq 2$, $4 \leq \bar{d} \leq 2n < 2n+4$. Thus,

$$\begin{aligned} LE(D_2(L_n)) &= \bar{d} + (2n-1)(\bar{d}-4) \\ &\quad + 3(2n-\bar{d}) \\ &\quad + (2n+4-\bar{d}) \\ &= \frac{8(2n^2-3n+2)}{n+2}. \end{aligned}$$

The only nonzero adjacency eigenvalues are $\pm 2\sqrt{2n}$. Hence, $D_2(L_n)$ is integral if and only if $2\sqrt{2n}$ is an integer. Let $2\sqrt{2n} = m$. Then $m^2 = 8n$. Since $8 \mid m^2$, it follows that $4 \mid m$. Writing $m = 4k$ gives $n = \frac{m^2}{8} = 2k^2$. Conversely, if $n = 2k^2$, then $2\sqrt{2n} = 2\sqrt{4k^2} = 4k$, which is an integer.

Therefore, $D_2(L_n)$ is integral precisely when $n = 2k^2$ for some positive integer k . ■

The complete bipartite representation explains the compact adjacency spectrum of $D_2(L_n)$: its energy is contributed entirely by one symmetric pair of nonzero eigenvalues.

B. The One-Thorn Graph of the Lintang Graph

Theorem 2. Let $n \geq 2$. The energy and Laplacian energy of the one-thorn graph $T(L_n)$ are

$$E(T(L_n)) = 2\sqrt{2n+4} + 2n$$

and

$$LE(T(L_n)) = \frac{4n}{n+2} + 2\sqrt{2}(n-1) + \sqrt{n^2+4} + \sqrt{n^2+4n+8},$$

respectively. Furthermore, $T(L_n)$ is neither integral nor Laplacian integral for any $n \geq 2$.

Proof. With the original vertices of L_n listed first and their pendant neighbors second,

$$A(T(L_n)) = \begin{pmatrix} A(L_n) & I_{n+2} \\ I_{n+2} & O_{n+2} \end{pmatrix}.$$

Hence,

$$\lambda I - A(T(L_n)) = \begin{pmatrix} \lambda I_{n+2} - A(L_n) & -I_{n+2} \\ -I_{n+2} & \lambda I_{n+2} \end{pmatrix}.$$

For $\lambda \neq 0$, applying the Schur complement to the lower-right block gives

$$\begin{aligned} \phi_A(T(L_n); \lambda) &= \lambda^{n+2} \det \left(\lambda I_{n+2} - A(L_n) - \frac{1}{\lambda} I_{n+2} \right) \\ &= \det \left(\lambda^2 I_{n+2} - \lambda A(L_n) - I_{n+2} \right). \end{aligned}$$

Both sides are polynomials in λ , so the identity extends to all λ . If λ_i is an adjacency eigenvalue of L_n , the corresponding eigenvalues λ of $T(L_n)$ satisfy $\lambda^2 - \lambda_i \lambda - 1 = 0$. Thus,

$$\lambda = \frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4}}{2}.$$

Since

$$\text{Spec}_A(L_n) = \left\{ \sqrt{2n}, -\sqrt{2n}, 0^{[n]} \right\},$$

define

$$\alpha = \frac{\sqrt{2n+4} + \sqrt{2n}}{2}$$

and

$$\beta = \frac{\sqrt{2n+4} - \sqrt{2n}}{2}.$$

Therefore,

$$\text{Spec}_A(T(L_n)) = \left\{ \alpha, -\alpha, \beta, -\beta, 1^{[n]}, (-1)^{[n]} \right\}.$$

Since $\alpha + \beta = \sqrt{2n+4}$, we obtain

$$\begin{aligned} E(T(L_n)) &= 2\alpha + 2\beta + 2n \\ &= 2\sqrt{2n+4} + 2n. \end{aligned}$$

For the Laplacian matrix,

$$L(T(L_n)) = \begin{pmatrix} L(L_n) + I_{n+2} & -I_{n+2} \\ -I_{n+2} & I_{n+2} \end{pmatrix}.$$

Thus,

$$\mu I - L(T(L_n)) = \begin{pmatrix} (\mu - 1)I_{n+2} - L(L_n) & I_{n+2} \\ I_{n+2} & (\mu - 1)I_{n+2} \end{pmatrix}.$$

For $\mu \neq 1$, the Schur complement yields

$$\phi_L(T(L_n); \mu) = \det \left(\mu(\mu - 2)I_{n+2} - (\mu - 1)L(L_n) \right).$$

Hence, if θ_i is a Laplacian eigenvalue of L_n , then

$$\mu^2 - (\theta_i + 2)\mu + \theta_i = 0,$$

and therefore

$$\mu = \frac{\theta_i + 2 \pm \sqrt{\theta_i^2 + 4}}{2}.$$

Since

$$\text{Spec}_L(L_n) = \left\{ 0, 2^{[n-1]}, n, n+2 \right\},$$

we obtain

$$\begin{aligned} \text{Spec}_L(T(L_n)) &= \{0, 2, \\ &\quad (2 + \sqrt{2})^{[n-1]}, \\ &\quad (2 - \sqrt{2})^{[n-1]}, \\ &\quad c_-, c_+, d_-, d_+\}, \end{aligned}$$

where

$$c_{\pm} = \frac{n+2 \pm \sqrt{n^2+4}}{2}$$

and

$$d_{\pm} = \frac{n+4 \pm \sqrt{n^2+4n+8}}{2}.$$

Moreover, $|V(T(L_n))| = 2n+4$ and $|E(T(L_n))| = 3n+2$. Hence,

$$\begin{aligned} \bar{d}(T(L_n)) &= \frac{2|E(T(L_n))|}{|V(T(L_n))|} \\ &= \frac{2(3n+2)}{2n+4} \\ &= \frac{3n+2}{n+2} \\ &= 3 - \frac{4}{n+2}. \end{aligned}$$

Write $\bar{d} = \bar{d}(T(L_n))$. Then $2 \leq \bar{d} < 3$. The Laplacian eigenvalues can now be located relative to $\bar{d}$. First, $2 - \sqrt{2} < 2 \leq \bar{d} < 3 < 2 + \sqrt{2}$. Moreover, $\sqrt{n^2+4} > n$ implies $c_- < 1$, $c_+ > n+1 \geq 3$, while $\sqrt{n^2+4n+8} \geq n+2$ implies $d_- < 1$, $d_+ > n+3$. Therefore, $c_- < \bar{d} < c_+$, $d_- < \bar{d} < d_+$. Consequently,

$$\begin{aligned} LE(T(L_n)) &= \bar{d} + (\bar{d} - 2) \\ &\quad + (n-1) \left[\bar{d} - (2 - \sqrt{2}) \right. \\ &\quad \left. + (2 + \sqrt{2} - \bar{d}) \right] \\ &\quad + (c_+ - c_-) + (d_+ - d_-) \\ &= 2\bar{d} - 2 + 2\sqrt{2}(n-1) \\ &\quad + \sqrt{n^2+4} \\ &\quad + \sqrt{n^2+4n+8}. \end{aligned}$$

Since $2\bar{d} - 2 = \frac{4n}{n+2}$, the stated formula follows. Suppose now that $T(L_n)$ were integral. Then α and β would both be integers. Since $\alpha + \beta = \sqrt{2n+4}$ and $\alpha - \beta = \sqrt{2n}$, both square roots would have to be integers. Let $2n = x^2$, $2n+4 = y^2$. Then $(y-x)(y+x) = 4$. The factors have the same parity, and the only nonnegative same-parity factorization of 4 gives $x = 0$, contradicting $n \geq 2$. Thus, $T(L_n)$ is not integral. Moreover, its Laplacian spectrum contains $2 + \sqrt{2}$ and $2 - \sqrt{2}$ with multiplicity $n-1 > 0$. Therefore, $T(L_n)$ is not Laplacian integral. ■

The thorn transformation retains a direct algebraic relation with the parent spectrum, but its quadratic pairing introduces irrational eigenvalues throughout the admissible family.

C. The Middle Graph of the Lintang Graph

Theorem 3. Let $n \geq 2$. The energy of $M(L_n)$ is

$$E(M(L_n)) = 2(n-1) + 2\sqrt{2}(n-1) + \sqrt{n^2+4} + \sqrt{n^2+4n+8},$$

while its Laplacian energy is

$$LE(M(L_n)) = \frac{(n+4)(n^2+3n-2)}{3n+2} + (n-1)\sqrt{n^2+8} + \sqrt{n^2-4n+16} + \sqrt{n^2-4n+8}.$$

Furthermore, $M(L_n)$ is neither integral nor Laplacian integral for any $n \geq 2$.

Proof. Let B be the vertex-edge incidence matrix introduced in Section III. With the original vertices listed before the edge-vertices,

$$A(M(L_n)) = \begin{pmatrix} O_{n+2} & B \\ B^T & B^T B - 2I_{2n} \end{pmatrix}.$$

Hence,

$$\lambda I - A(M(L_n)) = \begin{pmatrix} \lambda I_{n+2} & -B \\ -B^T & (\lambda+2)I_{2n} - B^T B \end{pmatrix}.$$

For $\lambda \neq 0$, the Schur complement gives

$$\phi_A(M(L_n); \lambda) = \lambda^{n+2} \det \left[(\lambda+2)I_{2n} - \frac{\lambda+1}{\lambda} B^T B \right].$$

For the 0-1 incidence matrix B , $BB^T = D(L_n) + A(L_n)$. Let $S = \text{diag}(I_2, -I_n)$. Since L_n is bipartite, $S L(L_n) S = D(L_n) + A(L_n) = BB^T$. Thus, BB^T is similar to $L(L_n)$. Moreover, BB^T and $B^T B$ have the same nonzero eigenvalues. Since

$$\text{Spec}_L(L_n) = \{0, 2^{[n-1]}, n, n+2\},$$

the spectrum of $B^T B$ is

$$\{0^{[n-1]}, 2^{[n-1]}, n, n+2\}.$$

Substitution into the determinant gives

$$\begin{aligned} \phi_A(M(L_n); \lambda) &= \lambda^{n+2} (\lambda+2)^{n-1} \\ &\quad \times \left(\frac{\lambda^2-2}{\lambda} \right)^{n-1} \\ &\quad \times \frac{\lambda^2 - (n-2)\lambda - n}{\lambda} \\ &\quad \times \frac{\lambda^2 - n\lambda - (n+2)}{\lambda}. \end{aligned}$$

After cancellation,

$$\begin{aligned} \phi_A(M(L_n); \lambda) &= \lambda(\lambda+2)^{n-1} \\ &\quad \times (\lambda^2-2)^{n-1} \\ &\quad \times (\lambda^2 - (n-2)\lambda - n) \\ &\quad \times (\lambda^2 - n\lambda - (n+2)). \end{aligned}$$

Since both sides are polynomials in λ and agree for $\lambda \neq 0$, the factorization also holds at $\lambda = 0$. Therefore,

$$\begin{aligned} \text{Spec}_A(M(L_n)) &= \{0, (-2)^{[n-1]}, \\ &\quad (\sqrt{2})^{[n-1]}, (-\sqrt{2})^{[n-1]}, \\ &\quad a_-, a_+, b_-, b_+\}, \end{aligned}$$

where

$$a_{\pm} = \frac{n-2 \pm \sqrt{n^2+4}}{2}$$

and

$$b_{\pm} = \frac{n \pm \sqrt{n^2+4n+8}}{2}.$$

For a quadratic $x^2 - px - q = 0$, $q > 0$, the product of the roots is $-q < 0$, so the roots have opposite signs. If $x_+ > 0 > x_-$, then $|x_+| + |x_-| = x_+ - x_- = \sqrt{p^2 + 4q}$. Hence, $|a_+| + |a_-| = \sqrt{n^2+4}$ and $|b_+| + |b_-| = \sqrt{n^2+4n+8}$. Thus,

$$\begin{aligned} E(M(L_n)) &= 2(n-1) + 2\sqrt{2}(n-1) \\ &\quad + \sqrt{n^2+4} \\ &\quad + \sqrt{n^2+4n+8}. \end{aligned}$$

We next determine the Laplacian spectrum. From Section III,

$$L(M(L_n)) = \begin{pmatrix} D(L_n) & -B \\ -B^T & (n+4)I_{2n} - B^T B \end{pmatrix}.$$

Consider the five natural vertex classes

$$\begin{aligned} &\{u_1\}, \quad \{u_2\}, \quad \{v_1, \dots, v_n\}, \\ &\{u_1 v_1, \dots, u_1 v_n\}, \quad \{u_2 v_1, \dots, u_2 v_n\}. \end{aligned}$$

Let $z \in \mathbb{R}^n$ satisfy $\mathbf{1}_n^T z = 0$. Consider vectors whose coordinates are zero at u_1 and u_2 , while the coordinates on each of the three n -vertex classes are scalar multiples of z . These vectors form an invariant subspace of $L(M(L_n))$. With respect to the three corresponding scalar coefficients, the restriction of $L(M(L_n))$ to this subspace is represented by

$$R = \begin{pmatrix} 2 & -1 & -1 \\ -1 & n+3 & -1 \\ -1 & -1 & n+3 \end{pmatrix}.$$

Its characteristic polynomial is

$$\det(\mu I_3 - R) = (\mu - n - 4) \times (\mu^2 - (n + 4)\mu + 2n + 2).$$

Because the subspace $\{z \in \mathbb{R}^n : \mathbf{1}_n^T z = 0\}$ has dimension $n - 1$, each eigenvalue of R contributes with multiplicity $n - 1$ to the Laplacian spectrum of $M(L_n)$. On the complementary class-constant subspace, the restriction is represented by

$$H = \begin{pmatrix} n & 0 & 0 & -n & 0 \\ 0 & n & 0 & 0 & -n \\ 0 & 0 & 2 & -1 & -1 \\ -1 & 0 & -1 & 3 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{pmatrix}.$$

A direct determinant calculation yields

$$\det(\mu I_5 - H) = \mu(\mu^2 - (n + 4)\mu + 3n) \times (\mu^2 - (n + 4)\mu + 3n + 2).$$

Since $3(n - 1) + 5 = 3n + 2 = |V(M(L_n))|$, these invariant components exhaust the full vertex space. Therefore,

$$\begin{aligned} \phi_L(M(L_n); \mu) &= \mu(\mu - n - 4)^{n-1} \\ &\quad \times (\mu^2 - (n + 4)\mu + 2n + 2)^{n-1} \\ &\quad \times (\mu^2 - (n + 4)\mu + 3n) \\ &\quad \times (\mu^2 - (n + 4)\mu + 3n + 2). \end{aligned}$$

Define

$$\begin{aligned} r_{\pm} &= \frac{n + 4 \pm \sqrt{n^2 + 8}}{2}, \\ s_{\pm} &= \frac{n + 4 \pm \sqrt{n^2 - 4n + 16}}{2}, \end{aligned}$$

and

$$t_{\pm} = \frac{n + 4 \pm \sqrt{n^2 - 4n + 8}}{2}.$$

Hence,

$$\begin{aligned} \text{Spec}_L(M(L_n)) &= \{0, (n + 4)^{[n-1]}, \\ &\quad r_-^{[n-1]}, r_+^{[n-1]}, \\ &\quad s_-, s_+, t_-, t_+\}. \end{aligned}$$

From Section II, $|V(M(L_n))| = 3n + 2$ and $|E(M(L_n))| = n^2 + 4n$. Thus,

$$\begin{aligned} \bar{d}(M(L_n)) &= \frac{2|E(M(L_n))|}{|V(M(L_n))|} \\ &= \frac{2(n^2 + 4n)}{3n + 2} \\ &= \frac{2n(n + 4)}{3n + 2}. \end{aligned}$$

Write $\bar{d} = \bar{d}(M(L_n))$. For $n \geq 2$,

$$\bar{d} - 3 = \frac{(2n + 3)(n - 2)}{3n + 2} \geq 0$$

and

$$n + 1 - \bar{d} = \frac{(n - 1)(n - 2)}{3n + 2} \geq 0.$$

Thus, $3 \leq \bar{d} \leq n + 1$. Since $\sqrt{n^2 + 8} > n$, we have $r_- < 2 < \bar{d} < r_+$. Similarly, $\sqrt{n^2 - 4n + 16} > n - 2$

and $\sqrt{n^2 - 4n + 8} > n - 2$ give $s_- < 3 \leq \bar{d} < s_+$ and $t_- < 3 \leq \bar{d} < t_+$. Also, $\bar{d} < n + 4$. Hence,

$$\begin{aligned} LE(M(L_n)) &= \bar{d} + (n - 1)(n + 4 - \bar{d}) \\ &\quad + (n - 1)(r_+ - r_-) \\ &\quad + (s_+ - s_-) \\ &\quad + (t_+ - t_-) \\ &= \frac{(n + 4)(n^2 + 3n - 2)}{3n + 2} \\ &\quad + (n - 1)\sqrt{n^2 + 8} \\ &\quad + \sqrt{n^2 - 4n + 16} \\ &\quad + \sqrt{n^2 - 4n + 8}. \end{aligned}$$

Finally, $\text{Spec}_A(M(L_n))$ contains $\pm\sqrt{2}$ with multiplicity $n - 1 > 0$. Thus, $M(L_n)$ is not integral. If $M(L_n)$ were Laplacian integral, then $n^2 + 8 = m^2$ for some integer m . Hence, $(m - n)(m + n) = 8$. Both factors have the same parity. The only positive same-parity factorization of 8 is $8 = 2 \cdot 4$, which gives $m - n = 2$, $m + n = 4$, and hence $n = 1$, outside the admissible range. Therefore, $M(L_n)$ is not Laplacian integral. ■

The middle transformation creates more spectral classes than the shadow and thorn constructions. Its incidence structure produces irrational eigenvalues throughout the family and a markedly different Laplacian-energy growth profile.

D. Numerical, Growth-Rate, and Arithmetic Comparison

Tables I and II provide representative numerical values for the three transformed graph families. These values are included only to illustrate the closed-form expressions. The general ordering and growth patterns discussed below are established analytically and are not inferred from the four numerical cases alone.

TABLE I
ENERGY OF THE TRANSFORMED LINTANG GRAPHS.

Graph	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$D_2(L_n)$	8.0000	9.7980	11.3137	12.6491
$T(L_n)$	9.6569	12.3246	14.9282	17.4833
$M(L_n)$	12.1290	18.6476	25.2820	31.9790

TABLE II
LAPLACIAN ENERGY OF THE TRANSFORMED LINTANG GRAPHS.

Graph	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$D_2(L_n)$	8.0000	17.6000	29.3333	42.2857
$T(L_n)$	12.1290	17.0476	21.9486	26.8361
$M(L_n)$	14.9282	24.2696	36.3825	51.2840

The energy values in Table I suggest the ordering

$$E(D_2(L_n)) < E(T(L_n)) < E(M(L_n)).$$

This ordering in fact holds for every $n \geq 2$. First,

$$\begin{aligned} E(T(L_n)) - E(D_2(L_n)) &= 2n + 2\sqrt{2n + 4} \\ &\quad - 4\sqrt{2n} \\ &> 2n - 2\sqrt{2n} \\ &\geq 0, \end{aligned}$$

where the last inequality follows from $n \geq 2$. Hence, $E(D_2(L_n)) < E(T(L_n))$ for every $n \geq 2$. It remains to compare $T(L_n)$ and $M(L_n)$. Since $\sqrt{n^2 + 4} > n$ and $\sqrt{n^2 + 4n + 8} > n + 2$, we obtain

$$\begin{aligned} E(M(L_n)) - E(T(L_n)) &> 2n + 2\sqrt{2}(n-1) \\ &\quad - 2\sqrt{2n+4} \\ &= 2 \left[n + \sqrt{2}(n-1) - \sqrt{2n+4} \right]. \end{aligned}$$

Thus, it is sufficient to show that $n + \sqrt{2}(n-1) > \sqrt{2n+4}$. Define $F(n) = (n + \sqrt{2}(n-1))^2 - (2n+4)$. Expanding gives

$$F(n) = n^2 + 2n\sqrt{2}(n-1) + 2(n-1)^2 - 2n - 4.$$

For $n \geq 2$,

$$n^2 \geq 4, \quad 2n\sqrt{2}(n-1) \geq 4\sqrt{2},$$

and

$$2(n-1)^2 - 2n - 4 = -6 + 2(n-1)(n-2) \geq -6.$$

Therefore, $F(n) \geq 4 + 4\sqrt{2} - 6 = 4\sqrt{2} - 2 > 0$. Since both $n + \sqrt{2}(n-1)$ and $\sqrt{2n+4}$ are positive, it follows that $n + \sqrt{2}(n-1) > \sqrt{2n+4}$. Consequently, $E(T(L_n)) < E(M(L_n))$. Combining the two inequalities yields $E(D_2(L_n)) < E(T(L_n)) < E(M(L_n))$ for every $n \geq 2$.

The closed-form expressions also reveal distinct energy-growth patterns. For the shadow graph,

$$\frac{E(D_2(L_n))}{\sqrt{n}} = 4\sqrt{2},$$

so $E(D_2(L_n))$ grows proportionally to $\sqrt{n}$. For the one-thorn graph,

$$\frac{E(T(L_n))}{n} = 2 + \frac{2\sqrt{2n+4}}{n}.$$

As n becomes arbitrarily large, the second term becomes negligible, and the ratio $E(T(L_n))/n$ approaches 2.

Similarly, dividing the energy of the middle graph by n gives

$$\begin{aligned} \frac{E(M(L_n))}{n} &= 2 \left(1 - \frac{1}{n} \right) + 2\sqrt{2} \left(1 - \frac{1}{n} \right) \\ &\quad + \sqrt{1 + \frac{4}{n^2}} + \sqrt{1 + \frac{4}{n} + \frac{8}{n^2}}. \end{aligned}$$

As n increases, this ratio approaches $4 + 2\sqrt{2}$. Hence, $E(T(L_n))$ and $E(M(L_n))$ grow linearly with n , whereas $E(D_2(L_n))$ grows proportionally to $\sqrt{n}$. These growth patterns follow directly from the closed-form expressions and are independent of the finite numerical sample in Table I. The Laplacian-energy comparison exhibits a different pattern. At $n = 2$, $LE(T(L_2)) > LE(D_2(L_2))$, whereas the ordering reverses from $n = 3$ onward. To establish this analytically, extend the two closed-form expressions to real $x \geq 3$ and define

$$\Delta(x) = LE(D_2(L_x)) - LE(T(L_x)).$$

The shadow formula can be rewritten as

$$LE(D_2(L_x)) = 16x - 56 + \frac{128}{x+2}.$$

Differentiating $\Delta(x)$ gives

$$\begin{aligned} \Delta'(x) &= 16 - \frac{136}{(x+2)^2} - 2\sqrt{2} \\ &\quad - \frac{x}{\sqrt{x^2+4}} - \frac{x+2}{\sqrt{x^2+4x+8}}. \end{aligned}$$

Each of the last two fractions is strictly less than 1. Hence, for $x \geq 3$,

$$\begin{aligned} \Delta'(x) &> 14 - 2\sqrt{2} - \frac{136}{(x+2)^2} \\ &\geq 14 - 2\sqrt{2} - \frac{136}{25} \\ &> 0. \end{aligned}$$

Thus, $\Delta(x)$ is strictly increasing for $x \geq 3$. At the starting point,

$$\begin{aligned} \Delta(3) &= \frac{76}{5} - 4\sqrt{2} - \sqrt{13} - \sqrt{29} \\ &\approx 0.5524 > 0. \end{aligned}$$

It follows that $LE(D_2(L_n)) > LE(T(L_n))$ for every integer $n \geq 3$, and the difference between the two Laplacian energies increases as n increases. The closed-form expressions also distinguish the corresponding Laplacian-energy growth rates. For the shadow graph,

$$\frac{LE(D_2(L_n))}{n} = \frac{8(2n^2 - 3n + 2)}{n(n+2)},$$

which approaches 16 as n becomes arbitrarily large. Thus, $LE(D_2(L_n))$ grows linearly with n .

For the one-thorn graph, dividing its closed-form expression by n shows that

$$\frac{LE(T(L_n))}{n}$$

approaches $2 + 2\sqrt{2}$ as n increases without bound. Hence, its Laplacian energy also grows linearly, but at a smaller limiting rate than that of the shadow graph.

In contrast, dividing the middle-graph expression by n^2 shows that

$$\frac{LE(M(L_n))}{n^2}$$

approaches $\frac{4}{3}$. Therefore, $LE(M(L_n))$ grows quadratically, whereas $LE(D_2(L_n))$ and $LE(T(L_n))$ grow linearly. This difference follows from the exact formulas rather than from the numerical values reported in Table II.

The arithmetic behavior gives a further distinction among the three families. The shadow graph is Laplacian integral for every $n \geq 2$, while its adjacency spectrum is integral precisely for the infinite subfamily determined by $n = 2k^2$. In contrast, the one-thorn and middle graphs contain irrational adjacency and Laplacian eigenvalues for every $n \geq 2$. Taken together, the exact inequalities, growth-rate analysis, and integrality results show that neighborhood duplication, pendant extension, and incidence transformation affect the same parent graph in fundamentally different spectral ways.

V. CONCLUSIONS

This study presents a unified spectral analysis of the shadow, one-thorn, and middle transformations of the lintang graph $L_n \cong K_{2,n}$. Exact expressions were obtained for the energy and Laplacian energy of $D_2(L_n)$, $T(L_n)$, and $M(L_n)$, together with their adjacency and Laplacian integrality properties.

The shadow graph retains the complete bipartite structure $D_2(L_n) \cong K_{4,2n}$ and is Laplacian integral for every $n \geq 2$, while its adjacency spectrum is integral precisely when $n = 2k^2$ for some positive integer k . In contrast, both $T(L_n)$ and $M(L_n)$ contain irrational adjacency and Laplacian eigenvalues for every $n \geq 2$, so neither family is integral nor Laplacian integral.

The three transformations leave clearly different spectral signatures on the same parent graph. The shadow construction preserves a compact adjacency spectrum, remains Laplacian integral throughout the admissible range, and admits an infinite integral subfamily. The one-thorn transformation produces paired eigenvalues through quadratic spectral relations, whereas the middle transformation generates a richer collection of spectral classes through its incidence structure. Their energy growth also differs: the shadow energy grows proportionally to $\sqrt{n}$, while the one-thorn and middle energies grow linearly with n . For Laplacian energy, the shadow and one-thorn families exhibit linear growth, whereas the middle graph shows quadratic growth. These differences demonstrate that neighborhood duplication, pendant extension, and incidence transformation reorganize the spectral structure of the same parent graph in distinct ways. Such behavior is relevant to graph-based structural models in which changes in connectivity are interpreted through eigenvalue-based descriptors.

Several directions remain open for further study. One natural extension is to replace the one-thorn construction by a generalized thorn graph in which more than one pendant vertex is attached to each original vertex and to determine how the resulting spectra, energies, and integrality conditions depend on the attachment parameters. Related problems include iterated shadow constructions, other incidence-based transformations, and broader complete bipartite base graphs such as $K_{m,n}$. More generally, it would be valuable to characterize structural conditions under which a graph transformation preserves adjacency or Laplacian integrality rather than introducing irrational eigenvalues.

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