Partition Dimension of Bridge Graphs Between Complete and Star Graphs

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***Abstract*—This paper of investigates the determination of the partition dimension for a *bridge graph* formed by connecting a clique K_n and a star $K_{1,m}$ through a single edge. Although the partition dimension has been extensively studied for various families and graph operations, the mixed dense–sparse case on $B(K_n, K_{1,m})$ remains unsettled, since the result is sensitive to the position of the bridge edge and the balance between the size parameters n and m. We combine distance symmetry arguments, leaf-based constraints at the star center, and explicit constructions of distinguishing partitions to obtain tight values of the partition dimension. The study begins with the basic cases K_1 and K_2, and then proceeds to the general case with parameters $n \geq 2$.**

The main result shows that for the *central bridge* ($e = v_1x$), it holds that $pd(B) = n - 1$ if $m < n$, $pd(B) = n$ if $m = n$, and $pd(B) = m$ if $m > n$; for the *leaf bridge* ($e = v_1u_1$), it holds that $pd(B) = n$ when $m \leq n$, and $pd(B) = m - 1$ when $m > n$. These results demonstrate that the location of the bridge edge, together with the size parameters m and n of the components, can sharpen the partition dimension value of the graph prior to the bridging operation.

I. INTRODUCTION

THE partition dimension refines the metric dimension. It asks for the smallest number of vertex classes whose distance profiles single out every vertex. The parameter measures how well a graph distinguishes its vertices and links directly to tasks in networks, navigation, and coding [8], [7].

Many papers have studied this topic on specific families. Examples include homogeneous firecrackers, cycle-book graphs, and several classes of trees [2], [21], [13], [6]. The parameter has also been analyzed under common graph operations such as corona products [1], [23], [12], subdivisions [3], Cartesian and strong products [14], and series–parallel constructions [18]. Related lines show how structure constrains distinguishability: bounds for wheel-like families [16], constant values in Toeplitz-type graphs [17], metric and connected-metric behavior in other families [20], [19], and labeling or thickness effects tied to distance and density [11], [9].

This paper focuses on a very small but informative change: adding a single *bridge edge* between two components. We join a complete graph K_n and a star $K_{1,m}$ with one edge uv where $u \in V(K_n)$ and $u \in V(K_{1,m})$, called a bridge graph $B(K_n, K_{1,m}, uv)$ [5], [22]. The construction mixes two extremes of connectivity: a dense clique and a leaf-heavy star. A bridge to the star center creates a cut-structure; a bridge to a leaf relaxes leaf constraints. Earlier studies in related settings gave upper bounds or partial cases [4], [5]. In the paper [5], only general upper and lower bounds for bridge graphs were identified. In contrast, the present study determines the *exact partition dimension* for bridge graphs constructed from a complete graph K_n and a star graph $K_{1,m}$, a specific case that was not addressed in the previous work. A uniform and exact description across sizes and across both bridge placements was not yet available.

We fill this gap. We give sharp formulas for the partition dimension of $B(K_n, K_{1,m})$. The value depends only on the balance between n and m and on whether the bridge hits the center or a leaf. Our method is constructive: distance-symmetry arguments yield lower bounds, and explicit partitions meet those bounds. A simple rule then emerges: the more restrictive side (the clique or the star leaves) controls the parameter, with at most a one-unit shift set by the bridge endpoint. These results extend and unify prior work on graph operations and bridge structures [1], [3], [5], [4], [14], [18], [23], [16].

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II. PRELIMINARIES

All graphs in this paper are finite, simple, and connected. Let $G = (V, E)$. A *partition* of V is a family $\Pi = \{T_1, \ldots, T_k\}$ of nonempty, pairwise disjoint sets with $\bigcup_{i=1}^{k} T_i = V$. For $u \in V$ and $X \subseteq V$, write $d(u, X) = \min\{d(u, x) : x \in X\}$ and define the *representation vector*

$$r(u \mid \Pi) = \big(d(u, T_1), \ldots, d(u, T_k)\big).$$

The partition Π is *distinguishing* if $r(u \mid \Pi) \neq r(v \mid \Pi)$ for all distinct $u, v \in V$. The *partition dimension* is

$$pd(G) = \min\{\, |\Pi| : \Pi \text{ is distinguishing} \,\} \quad \text{(see, e.g., [1], [20]).}$$

Note a basic property: if $u \in T_j$ then the j-th entry of $r(u \mid \Pi)$ equals 0, while the other entries are ≥ 1; for a clique K_n they are exactly 1.

Notation. We write K_n for the complete graph on $v_1, \ldots, v_n$, and $K_{1,m}$ for the star with center x and leaves $u_1, \ldots, u_m$. A *bridge graph* joining K_n and $K_{1,m}$ is obtained by adding a single edge e between a vertex of K_n and a vertex of $K_{1,m}$:

$$B(K_n, K_{1,m}; e), \qquad e \in \{v_1x,\ v_1u_1\}.$$

We call $e = v_1x$ the *center bridge* and $e = v_1u_1$ the *leaf bridge*. These conventions fix the distance calculations and partitions used later.

Three basic facts. We will use the following standard tools without reproving them.

Lemma II.1 ([14])**.** *Let G be connected and Π distinguishing. If $d(x, w) = d(y, w)$ for all $w \in V(G) \setminus \{x, y\}$, then x and y lie in different classes of Π.*

Lemma II.2 ([3])**.** *If a vertex of G is adjacent to k leaves, then $pd(G) \geq k$.*

Lemma II.3 ([17])**.** *For a connected graph G, we have $pd(G) = 2$ if and only if G is a path P_n.*

See also [15] for related examples that guide our constructions.

Immediate lower bounds. Let $G = B(K_n, K_{1,m}; e)$ with the bridge incident to $v_1 \in V(K_n)$. Then $v_2, \dots, v_n$ have the same distances to all remaining vertices, so by Lemma II.1 they must occupy distinct classes of any distinguishing partition:

$$pd(G) \;\geq\; n - 1. \tag{1}$$

If the bridge meets the star center x, Lemma II.2 forces all m leaves to be separated, hence

$$pd(G) \;\geq\; m. \tag{2}$$

Both estimates will become tight in the main results.

Proof roadmap. To match these bounds, we explicitly build small distinguishing partitions and check uniqueness of representation vectors in the three natural configurations for a pair (w, z): (i) both in K_n; (ii) both in $K_{1,m}$; (iii) one in each part (cf. [16]). When the bridge hits the center, we isolate the center (or one designated leaf) and split K_n accordingly, when it hits a leaf, we separate that leaf and adjust the classes inside K_n. The smallest valid construction gives an upper bound which, together with (1)–(2), yields the exact formulas one claimed later.

III. Results and Discussion

In this section we record exact values of the partition dimension for bridge graphs obtained from K_n and a star $K_{1,m}$. Throughout, $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ with $E(K_n) = \{v_i v_j | 1 \leq i \neq j \leq n\}$), and $V(K_{1,m}) = \{x, u_1, \dots, u_m\}$ where x is the center and $u_1, \dots, u_m$ are leaves.

A. Base cases for K_1 and K_2 joined to a star

We record four immediate propositions that anchor the small instances and guide the general proofs. These compact cases will serve as templates in what follows.

Proposition 1 (K_1 joined to $K_{1,1}$)**.** *For $B = B(K_1, K_{1,1}; e)$ one has $B \cong P_3$, hence $pd(B) = pd(P_3) = 2$.*

Proposition 2 (K_1 joined to $K_{1,m}$, $m \geq 2$:)**.** *If $B = B(K_1, K_{1,m}; v_1 x)$, then $B \cong K_{1,m+1}$ and therefore $pd\big(B(K_1, K_{1,m}; v_1 x)\big) = pd(K_{1,m+1}) = m + 1$[10].*

Proposition 3 (K_1 joined to $K_{1,2}$: bridge at a leaf)**.** *If $B = B(K_1, K_{1,2}; v_1 u_1)$, then $B \cong P_4$, hence $pd(B) = pd(P_4) = 2$.*

Proposition 4 (K_1 joined to $K_{1,3}$: bridge at a leaf)**.** *If $B = B(K_1, K_{1,3}; v_1 u_1)$, then $pd(B) = 3$. One distinguishing partition is $T_1 = \{u_1, x\}$, $\quad T_2 = \{u_2\}$, $\quad T_3 = \{u_3, v_1\}$.*

We now settle the remaining leaf bridge cases for K_1 joined to a star with $m \geq 4$. The $m - 1$ leaves adjacent to the center force at least $m-1$ classes, and a matching construction achieves this bound.

Lemma III.1 (K_1 joined to $K_{1,m}$, $m \geq 4$: bridge at a leaf)**.** *Let $B = B(K_1, K_{1,m}; v_1 u_1)$ with $m \geq 4$. Then $pd(B) = m - 1$.*

Proof. Since x is adjacent to $m - 1$ leaves $u_2, \dots, u_m$, Lemma II.2 gives $pd(B) \geq m - 1$. For the upper bound, consider

$$T_1 = \{u_1, x, u_2\},$$
$$T_2 = \{v_1, u_3\},$$
$$T_i = \{u_{\,i+1}\} \quad (3 \leq i \leq m - 1).$$

We check only pairs that lie in a common block.

Inside T_1. The pairs (u_1, x) and (u_2, x) are separated by $T_3 = \{u_4\}$, since $d(x, u_4) = 1$ while $d(u_1, u_4) = d(u_2, u_4) = 2$. The pair (u_1, u_2) is separated by T_2: $d(u_1, T_2) = 1$ and $d(u_2, T_2) = 2$.

Inside $T_2 = \{v_1, u_3\}$. Use T_1: $d(v_1, T_1) = 1$ (via u_1) while $d(u_3, T_1) = 1$ via x; to separate them pick any singleton $T_i = \{u_{i+1}\}$ with $i \geq 3$ (exists since $m \geq 4$): then $d(v_1, u_{i+1}) = 3$ and $d(u_3, u_{i+1}) = 2$.

All remaining T_i are singletons. Thus $\Pi = \{T_1, \dots, T_{m-1}\}$ is distinguishing, so $pd(B) \leq m - 1$. Together with the lower bound, $pd(B) = m - 1$. □

We next record five elementary properties for K_2 joined to a star. Taken together they pin down all small regimes and illustrate the center–vs.–leaf effect: a center bridge behaves like adding one leaf to the star (eventually forcing $pd = m$ when $m \geq 3$), while a leaf bridge keeps the value small at $m = 2$ (paths P_5) and stabilizes at $pd = 3$ for $m = 3$ with an explicit partition. These cases serve as templates for the general arguments. In Propositions 5 through 9, the proofs are straightforward, relying on the known facts that $pd(P_n) = 2$ [10] and Proposition 2, which uses $pd(K_{1,n}) = n$ [10].

Proposition 5 (K_2 joined to $K_{1,1}$)**.**
For $B = B(K_2, K_{1,1}; e)$ one has $B \cong P_4$, hence $pd(B) = pd(P_4) = 2$.

Proposition 6 (K_2 joined to $K_{1,2}$: bridge at a leaf)**.** *If $B = B(K_2, K_{1,2}; v_1 u_1)$, then $B \cong P_5$, hence $pd(B) = pd(P_5) = 2$.*

Proposition 7 (K_2 joined to $K_{1,2}$: bridge at the center)**.** *If $B = B(K_2, K_{1,2}; v_1 x)$, then $B \cong B(K_1, K_{1,3}; v_1 u_1)$ and $pd\big(B(K_2, K_{1,2}; v_1 x)\big) = pd\big(B(K_1, K_{1,3}; v_1 u_1)\big) = 3$.*

Proposition 8 (K_2 joined to $K_{1,m}$:bridge at the center)**.** *If $B = B(K_2, K_{1,m}; v_1 x)$ with $m \geq 3$, then $B \cong B(K_1, K_{1,m+1}; v_1 u_1)$. By Lemma III.1 (applied to $m+1 \geq 4$) we obtain $pd(B) = m$.*

Proposition 9 (K_2 joined to $K_{1,3}$: bridge at a leaf)**.** *If $B = B(K_2, K_{1,3}; v_1 u_1)$, then $pd(B) = 3$. One distinguishing partition is $T_1 = \{x, u_1, u_2\}$, $\quad T_2 = \{u_3\}$, $\quad T_3 = \{v_1, v_2\}$.*

The following lemma completes the leaf bridge side for K_2 when the star has at least four leaves: the $m-1$ leaves adjacent to the center impose a matching lower bound, and an explicit partition achieves it.

Lemma III.2 (K_2 joined to $K_{1,m}$, $m \geq 4$: bridge at a leaf)**.** *Let $B = B(K_2, K_{1,m};\, v_1u_1)$ with $m \geq 4$. Then $pd(B) = m-1$.*

Proof. Since x is adjacent to $m-1$ leaves $u_2, \dots, u_m$, Lemma II.2 gives $pd(B) \geq m-1$. For the upper bound, set

$$\begin{aligned} T_1 &= \{u_1, u_2, x\},\\ T_2 &= \{v_1, v_2, u_3\},\\ T_i &= \{u_{i+1}\} \quad (3 \leq i \leq m-1). \end{aligned}$$

Again we check pairs within blocks.

Inside T_1. The pairs (u_1, x) and (u_2, x) are separated by $T_3 = \{u_4\}$: $d(x, u_4) = 1$ while $d(u_1, u_4) = d(u_2, u_4) = 2$. The pair (u_1, u_2) is separated by T_2: $d(u_1, T_2) = 1$ and $d(u_2, T_2) = 2$.

Inside $T_2 = \{v_1, v_2, u_3\}$. Use T_1 to separate (v_1, v_2) and (v_2, u_3):

$$d(v_1, T_1) = 1, \quad d(v_2, T_1) = 2, \quad d(u_3, T_1) = 1.$$

For (v_1, u_3), take any singleton $T_i = \{u_{i+1}\}$ with $i \geq 3$ (exists since $m \geq 4$): then $d(v_1, u_{i+1}) = 3$ and $d(u_3, u_{i+1}) = 2$.

All remaining T_i are singletons. Hence $\Pi = \{T_1, \dots, T_{m-1}\}$ is distinguishing and $pd(B) \leq m-1$. Together with the lower bound, $pd(B) = m-1$. □

B. The bridge graph from K_n, $n \geq 3$ joined to $K_{1,m}, m \geq 3$

This subsection assembles the general picture for bridge graphs built from a clique K_n and a star $K_{1,m}$ when both parameters are at least 3. Two forces drive the outcome: the clique side enforces strong proximity among its vertices, while the star side forces many classes via its leaves. The position of the bridge at the star center or at a leaf modulates these forces by at most one. We first give a uniform upper bound (valid whenever $m \leq n$), then state three theorems that supply the exact values in the regimes $m \leq n$, $m > n$ with a leaf bridge, and $m > n$ with a center bridge.

The complete graph K_n and the star $K_{1,m}$ sit at opposite ends of connectivity. Gluing them with a single edge produces a mixed object: high density on the K_n side and degree–1 leaves on the star side. Our goal is to pin down the partition dimension by displaying small resolving partitions (upper bounds) and by invoking simple symmetry/leaf constraints (lower bounds).

When the star does not exceed the clique ($m \leq n$), one can always construct a resolving partition with exactly n classes, independently of whether the bridge meets the center or a leaf. This upper bound underpins the first exact result below.

Lemma III.3 (Upper bound for bridges with a small star)**.** *Let $K_{1,m}$ be a star, K_n be a complete graph and $B(K_n, K_{1,m}; e)$ be the bridge graph where $e \in \{v_1x,\ v_1u_1\}$. If $m \leq n$, then $B(K_n, K_{1,m}; e)$ admits a distinguishing partition with n classes. In particular, $pd\big(B(K_n, K_{1,m}; e)\big) \leq n$.*

Proof. We build a partition with n classes and inspect only unordered pairs that lie in the *same* class (pairs in different classes are already distinguished by the 0-entry in their own class).

Case $m = n$. Define $\Pi = \{T_1, \dots, T_n\}$ by

$$\begin{aligned} T_1 &= \{v_1, u_1, v_n\},\\ T_i &= \{v_i, u_i\}\ (2 \leq i \leq n-1),\\ T_n &= \{x, u_n\}. \end{aligned}$$

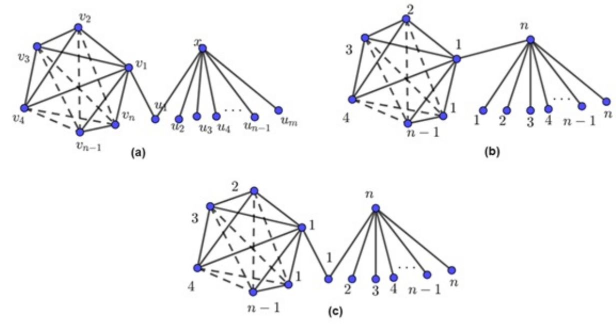

Fig. 1. (a) The bridge graph $B(K_n, K_{1,m};\, v_1u_1)$; (b) a distinguishing partition for the center bridge $e = v_1x$; (c) a distinguishing partition for the leaf bridge $e = v_1u_1$.

For visual guidance, Fig. 1(a) shows $B(K_n, K_{1,m};\, v_1u_1)$, Fig. 1(b) a partition for the center bridge $e = v_1x$, and Fig. 1(c) a partition for the leaf bridge $e = v_1u_1$.

Pairs inside T_1. The block T_n separates (v_1, v_n) and (u_1, v_n), since

$$d(v_1, T_n) = \begin{cases} 1, & e = v_1x,\\ 2, & e = v_1u_1, \end{cases}$$

$$d(v_n, T_n) = \begin{cases} 2, & e = v_1x,\\ 3, & e = v_1u_1, \end{cases}$$

$$d(u_1, T_n) = 1.$$

For the pair (v_1, u_1): use T_n when $e = v_1u_1$ (here $d(v_1, T_n) = 2 > d(u_1, T_n) = 1$); use T_2 when $e = v_1x$ (here $d(v_1, T_2) = 1 < d(u_1, T_2) = 2$).

Pairs inside $T_i = \{v_i, u_i\}$, $2 \leq i \leq n-1$. Use T_1: $d(v_i, T_1) = 1$ and $d(u_i, T_1) = 2$.

Pair inside $T_n = \{x, u_n\}$. Use T_1: $d(x, T_1) = 1$ and $d(u_n, T_1) = 2$.

Hence Π is distinguishing and $pd(G) \leq n$.

Case $m < n$. Define $\Pi = \{T_1, \dots, T_n\}$ by

$$\begin{aligned} T_i &= \{v_i, u_i\}\ (1 \leq i \leq m-1),\\ T_m &= \{x, u_m, v_m\},\\ T_j &= \{v_j\}\ (m+1 \leq j \leq n). \end{aligned}$$

For visual guidance, Fig. 2(a) illustrates the distinguishing partition when the bridge is $e = v_1x$ (center bridge), and Fig. 2(b) shows the partition when the bridge is $e = v_1u_1$ (leaf bridge).

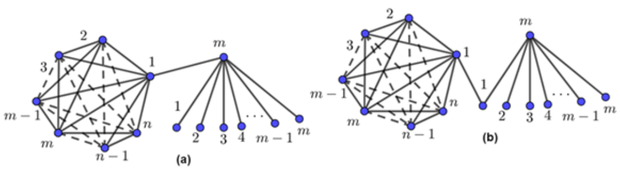

Fig. 2. The distinguishing partition for (a) the center bridge $e = v_1x$ and (b) the leaf bridge $e = v_1u_1$.

Pairs inside $T_i = \{v_i, u_i\}$, $1 \le i \le m-1$. Use $T_n = \{v_n\}$; since K_n is a clique,

$$d(v_i, T_n) = 1, \qquad d(u_i, T_n) = \begin{cases} 2, & e = v_1x, \\ 3, & e = v_1u_1. \end{cases}$$

Thus (v_i, u_i) is separated in either placement.

Pairs inside $T_m = \{x, u_m, v_m\}$. Use $T_1 = \{v_1, u_1\}$:

$$d(x, T_1) = 1, \qquad d(u_m, T_1) = 2, \qquad d(v_m, T_1) = 1,$$

which separates (x, u_m) and (u_m, v_m). The pair (x, v_m) is separated by $T_n = \{v_n\}$ since $d(v_m, T_n) = 1$ and $d(x, T_n) = 2$.

Singletons $T_j = \{v_j\}$, $m+1 \le j \le n$. No check is required.

Therefore every pair in a common class is distinguished, the partition Π is resolving, and $pd(G) \le n$. □

For $m \le n$, equal distance symmetry on the clique side forces n distinct classes among the vertices of K_n (aside from the bridge endpoint), yielding a matching lower bound. Together with Lemma III.3, this gives the exact value $pd = n$ for both bridge placements.

Theorem III.4. *Let $B = B(K_n, K_{1,m}; e)$ be the bridge graphs, K_n be a complete graph, and $K_{1,m}$ be a star graph, where $e \in \{v_1x,\ v_1u_1\}$. if $m \le n$, then $pd(B) = n$.*

Proof. By Lemma III.3 show the *upper bound* of graph B. In Lemma III.3 the graph B admits a distinguishing partition with n classes, hence $pd(B) \le n$.

Then, we show the *Lower bound* of graph B. By the equal distance lemma (Lemma II.1), the vertices $v_2, \dots, v_n$ must lie in pairwise distinct classes of any distinguishing partition (they have identical distances to all vertices outside $\{v_i, v_j\}$, for any distinct $i, j \ge 2$). Thus every distinguishing partition has at least $n-1$ classes.

If $|\Pi| = n-1$, then each class contains exactly one of $v_2, \dots, v_n$, so v_1 must share a class with some v_j ($2 \le j \le n$). Every other class contains a clique vertex v_k, and since K_n is complete graph we have $d(v_1, T) = 1 = d(v_j, T)$ for each such class T, while in their own class both distances are 0. Hence $r(v_1 \mid \Pi) = r(v_j \mid \Pi)$, a contradiction. Therefore no $(n-1)$-class distinguishing partition exists, so $pd(B) \ge n$.

Combining the bounds yields $pd(B) = n$. □

When the star is larger ($m > n$) and the bridge meets a *leaf*, the center remains adjacent to the other $m-1$ leaves, enforcing at least $m-1$ distinct classes. A direct construction achieves this bound, so the exact value is $m-1$.

Theorem III.5 (Leaf bridge, $m > n$)**.** *Let K_n be a complete graph, $K_{1,m}$ be a star graph, and $G = B(K_n, K_{1,m}; e)$ be the bridge graph with the bridge edge $e = v_1u_1$ (i.e., the bridge meets a leaf of the star). If $m > n$, then $pd(G) = m-1$.*

Proof. Lower bound. The center x of $K_{1,m}$ is adjacent to exactly $m-1$ leaves $u_2, \dots, u_m$. By Lemma II.2, any distinguishing partition must place these leaves in pairwise distinct classes; hence $pd(G) \ge m-1$.

Upper bound. Define a partition $\Pi = \{T_1, \dots, T_{m-1}\}$ of $V(G)$ by

$$\begin{aligned} T_1 &= \{v_1, v_n, u_2\}, \\ T_i &= \{u_{i+1},\, v_i\} \quad (2 \le i \le n-1), \\ T_i &= \{u_{i+1}\} \qquad (n \le i \le m-2), \\ T_{m-1} &= \{u_1, u_m, x\}. \end{aligned}$$

We verify that every unordered pair of vertices that lies in the *same* block is separated by at least one other block.

Block $T_1 = \{v_1, v_n, u_2\}$. The block T_{m-1} separates (v_1, v_n) and (v_n, u_2) since

$$d(v_n, T_{m-1}) = 2, \qquad d(v_1, T_{m-1}) = 1, \qquad d(u_2, T_{m-1}) = 1.$$

The pair (v_1, u_2) is separated by $T_2 = \{u_3, v_2\}$ because

$$d(v_1, T_2) = 1, \qquad d(u_2, T_2) = 2.$$

Blocks $T_i = \{u_{i+1}, v_i\}$ *for* $2 \le i \le n-1$. Use T_{m-1}:

$$d(u_{i+1}, T_{m-1}) = 1 \quad (\text{via } x), \qquad d(v_i, T_{m-1}) = 2 \quad (\text{via } u_1).$$

Blocks $T_i = \{u_{i+1}\}$ *for* $n \le i \le m-2$. Singletons; no check needed.

Block $T_{m-1} = \{u_1, u_m, x\}$. The block T_1 separates (u_1, u_m) and (x, u_m), since

$$d(u_1, T_1) = 1, \qquad d(x, T_1) = 1, \qquad d(u_m, T_1) = 2.$$

The pair (u_1, x) is separated by T_2:

$$d(x, T_2) = 1, \qquad d(u_1, T_2) = 2.$$

All within-block pairs are thus distinguished, so Π is a distinguishing partition with $|\Pi| = m-1$. Therefore $pd(G) \le m-1$. Combined with the lower bound, we conclude $pd(G) = m-1$. □

If $m > n$ and the bridge meets the *center*, all m leaves sit one step from the center, which forces at least m classes. An explicit m-class partition witnesses the upper bound, giving the exact value m.

Theorem III.6 (Exact values: $m > n$)**.** *Let $G = B(K_n, K_{1,m}; e)$ be the bridge graph obtained by joining K_n to $K_{1,m}$ with the bridge $e = v_1x$ (center bridge). If $m > n$ and $n \ge 3$, then $pd(G) = m$.*

Proof. Lower bound. The center x is adjacent to all m leaves $u_1, \dots, u_m$; hence, by Lemma II.2, $pd(G) \ge m$.

Upper bound. Define a partition $\Pi = \{T_1, \dots, T_m\}$ by

$$\begin{aligned} T_1 &= \{v_1, v_2, u_1\}, \\ T_i &= \{u_i,\, v_{i+1}\} \quad (2 \le i \le n-1), \\ T_i &= \{u_i\} \qquad (n \le i \le m-1), \\ T_m &= \{u_m, x\}. \end{aligned}$$

We only need to check unordered pairs that lie in the *same* block.

Inside $T_1 = \{v_1, v_2, u_1\}$. Use T_m to separate (v_1, v_2) and (v_2, u_1):

$$d(v_1, T_m) = 1, \quad d(v_2, T_m) = 2, \qquad d(u_1, T_m) = 1.$$

For (v_1, u_1), use $T_2 = \{u_2, v_3\}$:

$$d(v_1, T_2) = 1, \qquad d(u_1, T_2) = 2.$$

Inside $T_i = \{u_i, v_{i+1}\}$ *for* $2 \le i \le n-1$. Use T_m:

$$d(u_i, T_m) = 1, \qquad d(v_{i+1}, T_m) = 2.$$

Inside $T_i = \{u_i\}$ *for* $n \le i \le m-1$. Singletons; no check needed.

Inside $T_m = \{u_m, x\}$. Use T_1:

$$d(x, T_1) = 1, \qquad d(u_m, T_1) = 2.$$

Thus every within-block pair is distinguished by some block of Π, so Π is a resolving partition with $|\Pi| = m$, and hence $pd(G) \le m$. Together with the lower bound we obtain $pd(G) = m$. □

IV. Conclusion

We have shown that the partition dimension of a bridge graph obtained by joining a clique K_n and a star $K_{1,m}$ is tightly governed by the *structure* of the bridge (center vs. leaf) and by the *sizes* of the components. In particular, when the star does not exceed the clique ($m \le n$), a uniform construction yields $pd(B) = n$, confirming that the dense side dictates the parameter. For $m > n$, we have $pd(B) = m - 1$ when the bridge meets a *leaf* of the star ($e = v_1u_1$), and $pd(B) = m$ when it meets the *center* ($e = v_1x$).

V. Open problems.

How does the partition dimension of the graph change if the bridge edge is replaced by a path of length $t \ge 2$ (a subdivided bridge), or if several pairwise disjoint bridges are added.

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