

Connectivity of Power Graphs of Cyclic Groups with Specific Orders

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Abstract—The power graph is defined as a graph whose vertex set consists of the elements of a group, where two vertices are joined by an edge if one element is a power of the other. In this research we discuss the power graph of a cyclic group of order $n = pq$, with p and q distinct prime numbers, illustrates the connectivity among its vertices and reveals important topological properties such as the diameter and the clique number. Furthermore, this study also examines the degrees of the vertices and provides an analysis of the Hamiltonian, Eulerian, and planar characteristics of the graph. These findings highlight how algebraic structures can be effectively explored through graph-theoretical perspectives, offering deeper insights into the interplay between group theory and graph topology.

I. INTRODUCTION

GRAPH theory is a branch of mathematics that studies the definition, types, and properties of graphs. It provides a systematic framework for representing discrete objects and the relationships between them, allowing such structures to be visualized, analyzed, and interpreted in a more structured and intuitive manner. Through this representation, complex combinatorial and algebraic structures can often be simplified into graphical models that reveal hidden patterns and properties that are not immediately visible in their original form.

In recent decades, graph-theoretic approaches have been widely applied in various areas of mathematics, particularly in algebra. One of the most active research directions is the representation of algebraic structures, especially group theory, using graphs. This approach aims to bridge algebra and combinatorics by studying algebraic objects through their associated graphs, thereby providing deeper insights into their structure and behavior. As a result, many new concepts, definitions, and results have been established in this area, including coprime graphs [1], prime coprime graphs [2], and various related constructions.

Among these graph representations, one of the most extensively studied is the power graph of a group. The power graph construction has attracted significant attention due to its natural definition and strong connection with the internal structure of groups. It encodes algebraic relationships based on the power relation between group elements, making it a useful tool for analyzing subgroup structures and element orders.

The concept of the power graph of a finite group G , denoted by Γ_G , was first introduced by Kelarev and Quinn [3]. In their original formulation, the vertices of the graph correspond to the elements of G , and two distinct elements $a, b \in G$ are connected by a directed edge if $a^k = b$ for some positive integer k . Under this definition, the resulting graph is generally directed, since the power relation is not symmetric; that is, the existence of a path from a to b does not necessarily imply a path from b to a .

This concept was further extended in subsequent works, particularly in the context of semigroups [4], where similar combinatorial properties were investigated. Additional developments were also presented in [5], where various structural aspects of the resulting directed graphs were analyzed in more detail, particularly in relation to their algebraic and combinatorial properties.

Motivated by these developments, Chakrabarty *et al.* [6] introduced an undirected version of the power graph for semigroups and groups. In this modified version, two distinct vertices $a, b \in G$ are adjacent if and only if one is a power of the other, that is, $a^k = b$ or $b^k = a$ for some positive integer k . This modification removes the directionality of edges while preserving the essential algebraic structure encoded by the power relation. Consequently, the resulting graph is a simple undirected graph that is more suitable for classical graph-theoretic analysis.

Following this definition, various structural properties of power graphs have been extensively studied. These include connectivity, domination, independence number, vertex degree distribution, and other classical graph invariants. A number of extensions and generalizations of power graphs for different algebraic structures have also been explored in the literature [7], [8], [9]. In addition, power graphs of torsion-free groups and their structural properties have been investigated in [10], while connectivity-related results for specific classes of groups can be found in [11].

For cyclic groups, the structure of the power graph becomes particularly tractable due to the well-understood nature of their subgroup lattice. In this setting, many graph-theoretic properties can be explicitly determined. For instance, several studies have examined degree sequences and extremal properties of power graphs associated with cyclic groups [12]. Moreover, it has been shown that for the group of integers modulo a prime order n , the corresponding power graph is a complete graph K_n , with radius and diameter equal to one, while both the chromatic number and clique number attain the value n [13].

The present study extends these results by investigating the structure of power graphs associated with cyclic groups of

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order $n = pq$, where p and q are distinct prime numbers. In particular, we focus on several key graph-theoretic properties, including the diameter, clique number, Hamiltonian and Eulerian properties, as well as planarity. By combining algebraic properties of cyclic groups with classical results from graph theory, a more detailed characterization of the resulting power graphs is obtained.

II. METHODOLOGY

This research begins by analyzing the structure of the power graph formed over a cyclic group of order pq . Through this structural investigation, with particular emphasis on the degree of each vertex, several important graph-theoretic properties can be characterized. In particular, we derive results concerning the diameter, clique number, Hamiltonian property, Eulerian property, and planarity of the resulting graph.

To support the main analysis in a systematic manner, several preliminary definitions and fundamental concepts from group theory and graph theory are first presented. These foundational results serve as the basis for the subsequent discussions and are essential for establishing the main theorems in the later sections.

Definition 1. [14] A group G is said to be cyclic if there exists an element $a \in G$ such that

$$G = \{a^n \mid n \in \mathbb{Z}\}.$$

The element a is called a generator of G , and the cyclic group generated by a is denoted by $\langle a \rangle$.

Theorem 2. [14] Every cyclic group of order n is isomorphic to $\mathbb{Z}_n$.

Definition 3. [15] A connected graph G is called a Hamiltonian graph if it contains a cycle (a closed walk) that passes through every vertex of G exactly once. Such a cycle is called a Hamiltonian cycle.

Definition 4. [15] A connected multigraph G is called Eulerian if it contains a circuit that traverses every edge of G exactly once. Such a circuit is called an Eulerian circuit.

Theorem 5. [15] A connected multigraph G is Eulerian if and only if every vertex of G has even degree.

We also recall key auxiliary concepts such as the Euler totient function and the notion of clique number, which play an important role in analyzing the structural properties of power graphs. These preliminary definitions and theorems provide the necessary mathematical foundation for the development of the main results in the subsequent sections.

Definition 6. [15] For a positive integer n , the Euler totient function $\phi(n)$ is defined as the number of positive integers less than or equal to n that are relatively prime to n .

Corollary 7. [16] If $n = pq$, where p and q are distinct prime numbers, then

$$\phi(n) = (p-1)(q-1).$$

Theorem 8. [17] [18][Dirac's Theorem] Let G be a simple graph with $n \geq 3$ vertices. If

$$\deg(v) \geq \frac{n}{2}$$

for every vertex $v \in V(G)$, then G is Hamiltonian.

Theorem 9. [17] Let G be a simple planar graph with $n \geq 3$ vertices and e edges. Then,

$$e \leq 3n - 6.$$

Consequently, the average degree of G , denoted by $\bar{d}(G)$, satisfies

$$\bar{d}(G) = \frac{2e}{n} \leq \frac{6n - 12}{n} = 6 - \frac{12}{n} < 6.$$

In particular, every simple planar graph contains at least one vertex of degree at most 5.

Definition 10. [17] Let G be a simple graph. A clique in G is a subset of vertices in which every two distinct vertices are adjacent. The clique number of G , denoted by $\omega(G)$, is the size of the largest clique in G .

Definition 11. [17] Let G be a connected graph. The **diameter** of G , denoted by $\text{diam}(G)$, is the maximum distance between any pair of vertices in G . Formally,

$$\text{diam}(G) = \max_{u,v \in V(G)} d(u,v)$$

where $d(u,v)$ denotes the distance (the length of a shortest path) between vertices u and v .

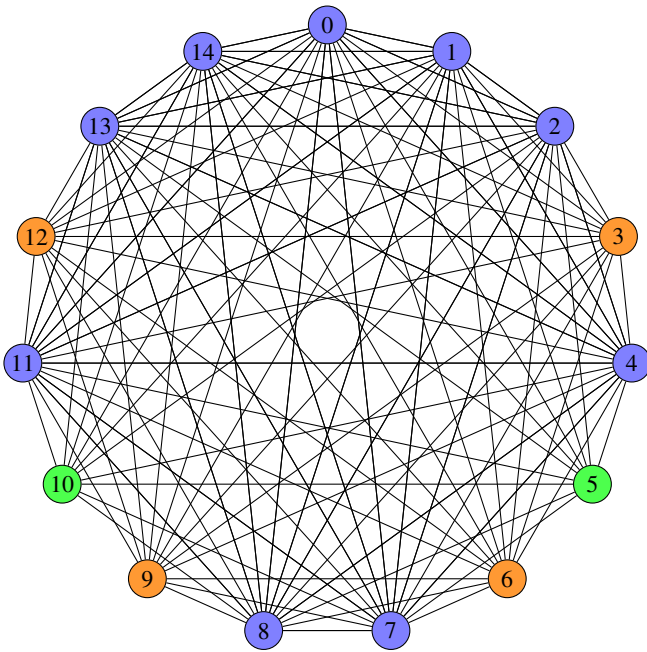
III. RESULT AND DISCUSSION

The discussion of power graphs over cyclic groups begins by revisiting the definition of the undirected version of the power graph.

Definition 12. [6] Let G be a group. The power graph of G is a simple graph in which all vertices are the elements of G . Two distinct vertices v and w are adjacent if $v^n = w$ or $w^m = v$ for some positive integers m and n . The power graph of the group G is denoted by Γ_G .

Based on Theorem 2, every cyclic group is isomorphic to the group $\mathbb{Z}_n$; therefore, the example below can serve as an illustration of the results concerning the power graph of a cyclic group.

Example 13. Given the group $\mathbb{Z}_{15}$, the corresponding power graph $\Gamma_{\mathbb{Z}_{15}}$ can be constructed in Figure 1.

Fig. 1. Power Graph Over Z_{15}

Based on the Figure 1, the adjacency matrices of this graph are present in below:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Therefore the degree of each vertex is as follows:

$$\begin{aligned} \deg(0) &= \deg(1) = \deg(2) = \deg(4) = \deg(7) = \deg(8) \\ &= \deg(11) = \deg(13) = \deg(14) = 14, \\ \deg(3) &= \deg(6) = \deg(9) = \deg(12) = 12, \\ \deg(5) &= \deg(10) = 10. \end{aligned}$$

It should be noted that the degrees of the vertices in power graph over Z_{15} can be grouped into three types. The identity vertex and the generators have degree 14, the vertices of order 5 have degree 12, while the vertices of order 3 have degree 10. In general, for a cyclic group of order pq , the set of vertices of order p is denoted by V_p , those of order q by V_q , and the generators by V_{pq} . Clearly,

$$|V_{pq}| = (p-1)(q-1), \quad |V_p| = q-1, \quad |V_q| = p-1.$$

Using this notation, the degrees of the vertices in the power graph are given in the following theorem.

The above structural pattern motivates a general classification of vertices in cyclic groups of composite order. In particular, for a cyclic group of order pq , where p and q are distinct primes, the vertices can be partitioned according to their orders. This partition plays a crucial role in determining the degree of each vertex in the corresponding power graph.

Theorem 14. Let G be a cyclic group of order $n = pq$, where p and q are distinct prime numbers. Then for every $v \in V(\Gamma_G)$,

$$\deg(v) = \begin{cases} pq-1, & v = e \text{ or } v \in V_{pq}, \\ (p-1)q, & v \in V_q, \\ (q-1)p, & v \in V_p. \end{cases}$$

Proof. Let G be a cyclic group of order pq . First, observe that the identity element e is adjacent to every element of G by definition of the power graph. Similarly, every generator of G generates the whole group, hence it is also adjacent to all other vertices. Therefore, for $v \in \{e\} \cup V_{pq}$, we obtain $\deg(v) = pq-1$.

Now consider $v \in V_p$. The element v generates a subgroup of order p . The only vertices not adjacent to v are the non-identity elements of the unique subgroup of order q , namely V_q . Since $|V_q| = q-1$, we have

$$\deg(v) = pq-1 - (q-1) = (p-1)q.$$

Similarly, for $v \in V_q$, the non-adjacent vertices are precisely the non-identity elements of the subgroup of order p , which has size $p-1$. Thus,

$$\deg(v) = pq-1 - (p-1) = (q-1)p.$$

Hence all cases are verified, and the theorem follows. $\square$

Having established the local structure of the graph through vertex degrees, we now turn our attention to its global connectivity properties. In particular, we investigate the diameter of the power graph, which measures the longest shortest path between any two vertices.

Theorem 15. Let G be a cyclic group of order pq , where p and q are distinct primes. Then the diameter of Γ_G is 2.

Proof. We show that every pair of distinct vertices is connected by a path of length at most 2. Let $u, v \in G$ be arbitrary vertices.

If u and v are adjacent, then the distance between them is 1. Otherwise, consider the identity element e and any generator $a \in V_{pq}$. By definition of the power graph, every element of G is adjacent to e , and every generator is adjacent to all elements of G .

Thus, even if u and v are not adjacent, there exists a path

$$u - e - v,$$

or

$$u - a - v,$$

with $a \in V_{pq}$. Hence the distance between any two vertices is at most 2, and so $\text{diam}(\Gamma_G) = 2$. $\square$

We next turn to the study of the clique number of the power graph, which measures the size of the largest complete subgraph in Γ_G . In the setting of cyclic groups, cliques arise naturally from chains of nested cyclic subgroups induced by the power relation. This structural connection allows us to characterize the clique number explicitly for groups of order pq , as stated in the following theorem.

Theorem 16. *Let p and q be distinct primes and let G be a cyclic group of order pq . Then*

$$\omega(\Gamma_G) = (p-1)(q-1) + \min(p-1, q-1) + 1.$$

Proof. A clique in a power graph corresponds to a set of elements that are mutually comparable via the power relation, which is equivalent to belonging to a chain of cyclic subgroups.

Consider the subgroup chains

$$\langle e \rangle \subset \langle a^q \rangle \subset \langle a \rangle, \quad \text{and} \quad \langle e \rangle \subset \langle a^p \rangle \subset \langle a \rangle.$$

The subgroup $\langle a \rangle$ has order pq and contains $(p-1)(q-1)$ generators. These generators, together with e , form a highly connected structure.

Extending this structure with either the p -subgroup or the q -subgroup yields two maximal cliques:

$$1 + (p-1) + (p-1)(q-1),$$

and

$$1 + (q-1) + (p-1)(q-1).$$

Taking the maximum of both expressions gives

$$\omega(\Gamma_G) = (p-1)(q-1) + \min(p-1, q-1) + 1.$$

This completes the proof. $\square$

Having characterized the clique number of Γ_G , we proceed to examine its global structural property, namely Hamiltonicity. This property reflects the existence of a cycle that traverses every vertex exactly once, and it is closely related to degree conditions in graphs. For the power graph of a cyclic group of order pq , Hamiltonicity can be derived using a well-known sufficient condition based on minimum degree, as presented in the following theorem.

Theorem 17. *Let p and q be distinct primes and let G be a cyclic group of order pq . Then Γ_G is Hamiltonian.*

Proof. From Theorem 14, the minimum degree of Γ_G is

$$\delta(\Gamma_G) = \min\{(p-1)q, (q-1)p\}.$$

We observe that

$$(p-1)q - \frac{pq}{2} = \frac{q(p-2)}{2} \geq 0, \quad (q-1)p - \frac{pq}{2} = \frac{p(q-2)}{2} \geq 0.$$

Thus,

$$\delta(\Gamma_G) \geq \frac{pq}{2}.$$

Since $|V(\Gamma_G)| = pq \geq 3$, by Dirac's Theorem, the graph Γ_G is Hamiltonian. $\square$

We now proceed to another classical topic in graph theory, namely Eulerian properties of graphs. Unlike Hamiltonicity,

which focuses on vertex traversal, Eulerianity is concerned with edge traversal and is determined by the parity of vertex degrees. In this part, we investigate whether the power graph of cyclic groups admits an Eulerian circuit under different orders of the group, particularly distinguishing between the cases of order $2p$ and pq , where p and q are primes.

Theorem 18. *Let G be a cyclic group of order $2p$, where p is an odd prime. Then Γ_G is not Eulerian.*

Proof. In a cyclic group of order $2p$, the identity element e is adjacent to all other vertices, so

$$\deg(e) = 2p - 1,$$

which is odd. Since an Eulerian graph requires all vertex degrees to be even, Γ_G cannot be Eulerian. $\square$

Theorem 19. *Let G be a cyclic group of order pq , where p and q are odd primes. Then Γ_G is Eulerian.*

Proof. From Theorem 3.3, the degree of every vertex is either $(p-1)q$, $(q-1)p$, or $pq-1$. Since p and q are odd, all these quantities are even.

Hence every vertex in Γ_G has even degree. Since the graph is connected, it follows that Γ_G is Eulerian. $\square$

Finally, we examine the planarity of the power graph. Planarity is a fundamental concept in graph theory that determines whether a graph can be embedded in the plane without edge crossings. In this part, we apply Euler's inequality for planar graphs to analyze the structural constraints of Γ_G . In particular, we show that the power graph of a cyclic group of order pq is planar only in a very restricted case.

Theorem 20. *Let p and q be distinct primes. Then Γ_G is planar if and only if $\{p, q\} = \{2, 3\}$.*

Proof. The total degree sum of the graph is

$$S = (1 + (p-1)(q-1))(pq-1) + (p-1)2q + (q-1)2p,$$

which simplifies to

$$S = p^2q^2 - 3pq + 2p + 2q - 2.$$

Hence the average degree is

$$\bar{d} = \frac{S}{pq} = pq - 3 + \frac{2p + 2q - 2}{pq}.$$

For $pq \geq 10$, we obtain $\bar{d} \geq 6$, which violates Euler's inequality for planar graphs. Thus the graph is non-planar in this case.

The only possible case satisfying planarity is $pq = 6$, i.e., $\{p, q\} = \{2, 3\}$. Therefore, Γ_G is planar if and only if $\{p, q\} = \{2, 3\}$. $\square$

IV. CONCLUSIONS

The power graph of a cyclic group of order $n = pq$, where p and q are distinct primes, always has a constant diameter of 2, and its clique number is precisely given by $\omega(\Gamma_G) = (p-1)(q-1) + \min(p-1, q-1) + 1$. Furthermore, this graph always contains a Hamiltonian path and is therefore classified as a Hamiltonian graph.

Regarding its advanced structural properties, the graph is never Eulerian if the cyclic group has order $n = 2p$ with p an odd prime, whereas it results in an Eulerian graph when both p and q are distinct odd primes. Additionally, planarity is highly restricted for this structure, being guaranteed if and only if $\{p, q\} = \{2, 3\}$.

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