Graphs with Large Locating Rainbow Vertex Connection Number

Tita Khalis Maryati[1,*] and Fawwaz Fakhrurrozi Hadiputra[2]

***Abstract*—The locating rainbow vertex connection number of G, denoted by $rvcl(G)$, is the least positive integer k such that there exists a k-coloring of G that ensures G is rainbow vertex-connected and each vertex has a distinct rainbow code. To learn how this parameter behave in graphs, it is natural to consider the extremal case. In this paper, we investigate several graphs with the difference of its order and its locating rainbow vertex connection number is not larger than three, i.e. $rvcl(G) \in \{n, n-1, n-2, n-3\}$ for graphs of order n. Furthermore, we demonstrate the existence of a graph with arbitrary large difference of its partition dimension and its locating rainbow vertex connection number. The study provides an overview of some structures that distinguishability under chromatic constraints becomes most costly, as in large number of colors needed.**

I. INTRODUCTION

LET G be a simple, undirected and finite graph. For a positive integer k, let φ be a function that maps the vertices of G to colors which are denoted by integers from 1 up to k inclusively. The map φ is said to be a *proper vertex coloring* if every image of two adjacent vertices from φ are distinct. Some applications of proper k-coloring are found in index registers, scheduling problems, and frequency assignments [19]. A natural variation of proper k-coloring would be the assignment of colors for edges instead of vertices, called *proper edge coloring*.

The concept of rainbow edge coloring emerged in the broader context of graph theory. Introduced by Chartrand *et al.* [7], the *rainbow connection number* is the minimum number of colors needed to color the edges of a graph so that every pair of vertices is connected by a rainbow path, that is a path where no two edges share the same color. This idea was inspired by earlier work on graph connectivity and edge coloring, such as proper edge coloring (where adjacent edges must have distinct colors) and the study of path connectivity in graphs. To date, the study of rainbow connection number is still continuing [1], [9].

Over time, Krivelevich and Yuster [15] considered its variation. For a positive integer k, a map from the vertices of G to integers from 1 up to k inclusively is called *rainbow vertex coloring* if for every two vertices u and v in G, there exists a path connecting u and v whose internal vertices have distinct colors. In this case, we say u and v is rainbow-connected. The smallest number k such that there exists a rainbow vertex coloring of G among all rainbow vertex coloring of G is said to be *rainbow vertex connection number* of G, which is denoted by $rvc(G)$. Some recent progress of rainbow vertex connection number can be seen in [8], [25].

Building on the idea of graph connectivity and coloring, another fascinating concept in graph theory is the metric dimension. The *metric dimension* of a graph is a measure of how uniquely the vertices of the graph can be identified based on their distances to a selected subset of vertices, called a resolving set. Formally, a resolving set is a subset of vertices S such that for every pair of distinct vertices u and v in the graph, there exists at least one vertex $s \in S$ for which the distance from s to u is different from the distance from s to v. The minimum cardinality of such a resolving set is called the metric dimension of the graph. This concept was first introduced in the 1970s by Harary and Melter [10], and later independently by Slater [20]. Lately, the study of metric dimension has been conducted on circulant graphs [23], amalgamation of theta graphs [24], the edge comb product of graphs [17], [18]. Applications of metric dimension can be seen in [16] and [21].

Moreover, there is another way to identify vertices using partition of vertices. While the metric dimension focuses on uniquely identifying vertices using distances to a resolving set, the partition dimension extends this idea by considering how vertices can be distinguished based on their distances to a partition of the vertex set of a graph. In formal way, the *partition dimension* of a graph, denoted by $pd(G)$, is the least integer k for which there exists a partition of the vertex set into k subsets (called resolving partitions) such that every pair of distinct vertices is distinguished by their distances to at least one subset in the partition. This concept was introduced by Chartrand *et al.* [6] as a generalization of the metric dimension. Tomescu [22] characterized non-isomorphic graphs G of order at least 7 with $pd(G) = |V(G)| - 2$. This result then got revisions given in [2], [12]. In addition, a characterization of non-isomorphic graphs G with $pd(G) = |V(G)| - 3$ is also determined [2], [12]. Furthermore, Haspika *et al.* [13] determined the partition dimension of a grid graph. In addition, Haryeni *et al.* [11] constructed several families of graphs with partition dimension three.

Recently, a variant which combines the notion of rainbow vertex coloring and partition dimension is introduced by Bustan *et al.* [3]. These partition captures the vertex distinguishability and rainbow-connection simultaneously, which gives rise

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to an application for establishing building security systems [4]. Let k be a positive integer and $\varphi : V(G) \to \{1, 2, \dots, k\}$ be a map. Let $\Pi = \{R_1, R_2, \dots, R_k\}$ be a partition of vertices to colors where $R_i = \{u \in V(G) \mid \varphi(u) = i\}$. A rainbow path in G is a path such that there are no internal vertices of the path that is contained in a same set R_i for some $i \in [1, k]$. The distance $d(u, R_i)$ is defined by the minimum of $d(u, r)$ for all $r \in R_i$. The *rainbow code* of a vertex u in G with respect to Π is defined by

$$(d(u, R_1), d(u, R_2), \dots, d(u, R_k)),$$

which is denoted by $rc_\Pi(v)$. The map φ is a *locating rainbow vertex coloring* of G if every pair of vertices in G is connected by a rainbow path and every vertex in G has a unique rainbow code. The *locating rainbow vertex connection number* of G, denoted by $rvcl(G)$, is the smallest number of colors taken over all locating rainbow vertex colorings of G. Bustan *et al.* [3] determined the locating rainbow vertex connection number of complete graphs, stars, and paths. A necessary condition of order of a graph G which has $rvcl(G) = r$ for some r has been determined in [3]. Moreover, a characterization of graphs G having $rvcl(G) = 2$ and $rvcl(G) = n - 1$ is determined in [4]. By the same authors, some bounds on $rvcl(G)$ where G is r-regular bipartite graph are presented. An exact value of $rvcl(G)$ where G is a complete bipartite graph with or without matching deletion is given in [4]. Further, Imrona *et al.* [14] considered the locating rainbow vertex coloring of certain comb product of graphs. Most lately, Bustan *et al.* [5] presented the locating rainbow vertex coloring of cycles C_n, $K_{2n} - nK_2$, and $K_n - C_n$.

Despite the number of study conducted, little is known about graphs with locating rainbow vertex connection number close to its order. Recognizing this is crucial, since there are some structural scenarios where enforcing distinguishability through chromatic constraints incurs the highest cost, reflected in the requirement of many colors. In this paper, we present several families of graphs G with $rvcl(G) = |V(G)| - i$ for $i \in \{1, 2, 3\}$. In addition, we provide some connections of locating rainbow vertex connection and partition dimension of graphs.

II. MAIN RESULTS

By definition, a locating rainbow vertex coloring ensure each pair of distinct vertices to be rainbow-connected. Hence, it is also a rainbow vertex coloring. Therefore, for any graph G, we have the following inequality:

$$rvc(G) \leq rvcl(G).$$

Likewise, a locating rainbow vertex coloring ensure each vertex to have a unique rainbow code (representations). Thus, it is also a resolving partition. Hence, the following also holds:

$$pd(G) \leq rvcl(G). \qquad \text{(II.1)}$$

In particular, we can say more if the graph has $diam(G) \leq 2$.

Proposition 1. *If G has $diam(G) \leq 2$, then $rvcl(G) = pd(G)$.*

Proof. If a graph has $diam(G) \leq 2$, then any pair of vertices in G is either adjacent or has a distance of 2. In any case, the shortest path between this pair will visit at most one other vertex. This implies that this pair of vertices is rainbow-connected. Thus, $rvcl(G) \leq pd(G)$. Combining with (II.1), we obtain $rvcl(G) = pd(G)$. □

Therefore, it is important to know several results about partition dimension. Consider the following properties of partition dimension which is shown by Chartrand *et al.* [6].

Proposition 2. *[6] Let G be a connected graph of order n. Then $pd(G) = n$ if and only if $G \cong K_n$.*

Theorem 3. *[6] Let G be a connected graph of order n. Then $pd(G) = n - 1$ if and only if one of the following holds:*

1) $G \cong K_{1,n-1}$,
2) $G \cong K_n - e$,
3) $G \cong K_1 + (K_1 \cup K_{n-2})$.

Then, the following characterization of graphs G having $rvcl(G) = |V(G)|$ is proved by Bustan *et al.* [4].

Theorem 4. *[4] Let G be a connected graph of order $n \geq 3$. Then $rvcl(G) = n$ if and only if $G \cong K_n$.*

This is one of case in which the locating rainbow vertex connection number coincides with the partition dimension. We can obtain more result for graphs with $pd(G) = |V(G)| - 1$.

Corollary 5. *Let G be a graph of order n. Then $rvcl(G) = n - 1$ if G is isomorphic to one of the following:*

1) $K_{1,n-1}$,
2) $K_n - e$,
3) $K_1 + (K_1 \cup K_{n-2})$.

Proof. Observe that each graph $K_{1,n-1}, K_n - e$, and $K_1 + (K_1 \cup K_{n-2})$ has the diameter of two, which implies $rvcl(G) = pd(G)$ when G is either $K_{1,n-1}, K_n - e, K_1 + (K_1 \cup K_{n-2})$ due to Proposition 1. □

Not every graph G has the same value of $rvcl(G)$ and $pd(G)$. In the following theorem, we are able to show that there exists a graph G for some desired value of $a, b \in \mathbb{Z}$ where $pd(G) = a$ and $rvcl(G) = b$. Let $\varphi : V(G) \to [1, k]$ for some positive integer k. For some integer $i \in [1, k]$, set $\varphi^{-1}(i) = \{v \in V(G) \mid \varphi(v) = i\}$. Now, to check the distinguishability of rainbow codes, it is sufficient to only consider pairs with the same color. Indeed, if we have a pair of vertices with distinct color then the 0 entry of each vertices will be positioned differently, which corresponds to each color.

Theorem 6. *Let G be a graph of order $n \geq 4$. Let a and b be any integers satisfying $2 \leq a < b \leq n - a$. Then there exists a graph with $pd(G) = a$ and $rvcl(G) = b$.*

Proof. Consider a graph $B_{a,b}$ given by

$$V(B_{a,b}) = \{u_i, v_j \mid i \in [1, a], j \in [0, b]\},$$
$$E(B_{a,b}) = \{u_i v_0, v_j v_{j+1} \mid i \in [1, a], j \in [0, b - 1]\}.$$

In this case, the graph $B_{a,b}$ has an order of $a+b+1$. Since $u_1, u_2, \ldots, u_a$ are twin vertices, then $pd(B_{a,b}) \geq a$. Now, define $\varphi : V(B_{a,b}) \to [1,a]$ where

$$\varphi(u_i) = i, \qquad \text{for } i \in [1,a],$$
$$\varphi(v_j) = \begin{cases} j+1, & \text{if } j \in \{0,1\}, \\ 3, & \text{if } j \in [2,b]. \end{cases}$$

It is clear that the representations of two v_i's are pairwise distinct. Consider u_i and v_j in the same partition.

- If $\varphi(u_i) = 1$, then $i = 1 = j$ such that $d(u_1, \varphi^{-1}(2)) = 2 > 1 = d(v_1, \varphi^{-1}(2))$.
- If $\varphi(u_i) = 2$, then $i = 2 = j$ such that $d(u_2, \varphi^{-1}(3)) = 2 > 1 = d(v_2, \varphi^{-1}(3))$.
- If $\varphi(u_i) = 3$, then $i = 3$ such that $d(u_3, \varphi^{-1}(1)) = 1 < j = d(v_j, \varphi^{-1}(1))$ for $j \in [2,b]$.

Then, every representations are unique which implies $pd(B_{a,b}) = a$. Moreover, observe that there are b internal vertices of $B_{a,b}$. Now, we will show that $rvcl(B_{a,b}) = b$. Let φ be a locating rainbow vertex coloring of G. Among $\{v_0, v_1, \ldots, v_{b-1}\}$, suppose that there exists v_i and v_j where $\varphi(v_i) = \varphi(v_j)$. Then, there is no rainbow path from u_1 to v_b. This implies $\{\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_{b-1})\}$ must be pairwise distinct. This shows $rvcl(B_{a,b}) \geq b$.

To show $rvcl(B_{a,b})$, define a map $\varphi : V(B_{a,b}) \to [1,b]$ given by

$$\varphi(u_i) = i, \qquad i \in [1,a],$$
$$\varphi(v_i) = \begin{cases} i+1, & i \in [0,b-1], \\ b, & i = b. \end{cases}$$

Observe that any path between two vertices of $B_{a,b}$ passes only $v_0, v_1, \ldots, v_{b-1}$. Since these vertices already colored pairwise distinct, it follows that $B_{a,b}$ is rainbow connected. Now, consider two vertices of $B_{a,b}$ which have the same color. Observe that

- for every $i \in [1,a]$, $d(v_{i-1}, v_{b-1}) = b - i$ and $d(u_i, v_{b-1}) = b$;
- $d(v_0, v_{b-1}) = b-1$ and $d(v_0, v_b) = b$.

This implies that φ is a locating vertex rainbow coloring of $B_{a,b}$ and thus $rvcl(B_{a,b}) = b$. □

We present an illustration of $B_{3,5}$ in Figure 1.

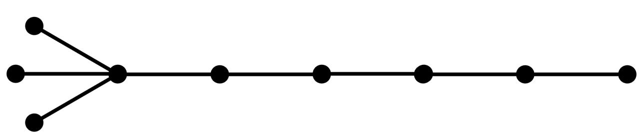

Fig. 1. The graph $B_{3,5}$ where $pd(B_{3,5}) = 3$ and $rvcl(B_{3,5}) = 5$.

Next, we continue to consider graphs G with $rvcl(G) = |V(G)| - 2$. Similarly, one possible way to achieve such graphs is to check the graphs having $pd(G) = |V(G)| - 2$. A characterization of graphs with $pd(G) = |V(G)| - 2$ is due Tomescu [22] along with some revisions given in [2], [12]. Following the notations given in [2], [12], [22], we define some special graphs as follows:

- F consists of $K_{n-1} - e$ and another vertex adjacent to only one of the endvertices of e;
- G_1 consists of $K_{n-1} - e$ and another vertex adjacent to the endvertices of e;
- G_2 consists of K_{n-1} and a vertex adjacent to two vertices of K_{n-1};
- $G_4 \cong (K_1 \cup K_2) + K_{n-3}$;
- G_7 consists of K_{n-2} and a path P_4 joining two vertices of K_{n-2};
- G_8 consists of K_{n-2} and K_3 having a common vertex;
- G_9 consists of K_{n-2} and P_3 having in common the central vertex of P_3;
- G_{10} consists of K_{n-2} and P_3 having in common the endvertex of P_3;
- G_{12} consists of K_{n-2} and a path P_4 having the central edge in common with K_{n-2}.

We give an illustration for F, G_1, G_2, G_7, G_8, G_9, and G_{10} in Figure 2. Tomescu [22] proved that $K_{2,n-2}$, $K_2 + \overline{K_{n-2}}$, $K_n - E(P_3)$, $K_n - E(K_3)$, $K_n - E(P_4)$, $K_n - E(C_4)$, $K_n - E(2K_2)$, $K_1 + (K_1 \cup (K_{n-2} - e))$, $(K_1 \cup K_2) + K_{n-3} \cong G_4$, F, G_1, G_2, G_7, G_8, G_9, G_{10}, and G_{12} to have partition dimension of $n-2$ where $n \geq 9$ is the order of each graph. Tomescu also present a proof that these graphs and some other graphs (we do not mention them here for clarity) are the only graphs with partition dimension of $n-2$. However, a revision came in [2], [12] where they determine that the graphs not mentioned here does not have a partition dimension of $n-2$. In addition, Baskoro and Haryeni [2] add the graph F to be among graphs whose partition dimension is $n-2$. In the following theorem, we will only consider graphs whose order is at least 9, since this is a condition provided by [22] which we rely on.

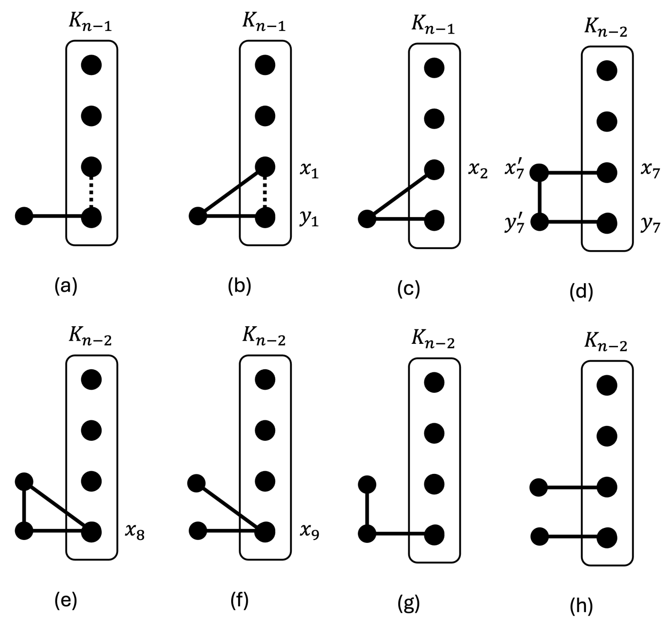

Fig. 2. The graphs (a) F, (b) G_1, (c) G_2, (d) G_7, (e) G_8, (f) G_9, (g) G_{10} and (h) G_{12} where dotted lines means non-adjacency.

Theorem 7. *Let G be a graph of order $n \geq 9$. If $pd(G) = n-2$, then $rvcl(G) = n-2$.*

Proof. If a graph has $pd(G) = n - 2$ where $n \geq 9$, then $rvcl(G) \geq n-2$. From a result given in [22], [2], [12], graphs G with $pd(G) = n - 2$ must be isomorphic to one of the following: $K_{2,n-2}$, $K_2 + \overline{K_{n-2}}$, $K_n - E(P_3)$, $K_n - E(K_3)$, $K_n - E(P_4)$, $K_n - E(C_4)$, $K_n - E(2K_2)$, $K_1 + (K_1 \cup (K_{n-2} - e)$, $(K_1 \cup K_2) + K_{n-3} \cong G_4$, F, G_1, G_2, G_7, G_8, G_9, G_{10}, or G_{12}.

First, we will show that many of these graphs inherited the parameters based on the diameter. In particular, we will show that other than F, G_{10}, or G_{12} these graphs have a diameter of 2. A join product of graph must always has a diameter of 2. Therefore, the graphs $K_{2,n-2} \cong \overline{K_2} + \overline{K_{n-2}}$, $K_2 + \overline{K_{n-2}}$, $K_1 + (K_1 \cup (K_{n-2} - e)$, and $(K_1 \cup K_2) + K_{n-3} \cong G_4$ have a diameter of 2. Moreover, a deletion of edges less than $\frac{n}{2}$ in K_n yields a vertex that has no incident edges deleted. This implies such edge deletion does not remove the property of a graph having a dominating vertex. Therefore, $K_n - E(P_3)$, $K_n - E(K_3)$, $K_n - E(P_4)$, $K_n - E(C_4)$, and $K_n - E(2K_2)$ have the diameter of 2. In addition, for $i \in \{2, 8, 9\}$ the vertex $x_i \in V(G_i)$ shown in Figure 2 is a dominating vertex of G_i. Therefore, $diam(G_i) = 2$ for $i \in \{2, 8, 9\}$. In particular, consider $x_1, y_1 \in V(G_1)$ shown in Figure 2(b) where $N_{G_1}(x_1) = V(G_1) \setminus \{y_1\}$ and $N_{G_1}(y_1) = V(G_1) \setminus \{x_1\}$. Since $d_{G_1}(x_1, y_1) = 2$, then $diam(G_1) = 2$. Lastly, consider $x_7, y_7, x_7', y_7' \in V(G_7)$ shown in Figure 2(d) where $N_{G_7}(x_7) = V(G_7) \setminus \{y_7'\}$ and $N_{G_7}(y_7) = V(G_7) \setminus \{x_7'\}$. Since $d(x_7', y_7') = 1$, then $diam(G_7) = 2$.

Since a graph with $diam(G) = 2$ has $pd(G) = rvcl(G)$, then we are left to consider F, G_{10} and G_{12}. Let F be a graph obtained by

$$\begin{aligned} V(F) &= \{u\} \cup \{v_i \mid i \in [1, n-1]\}, \\ E(F) &= \{uv_{n-1}\} \cup \{v_i v_j \mid i, j \in [1, n-1], \\ &\qquad i \neq j\} \setminus \{v_{n-2}v_{n-1}\}. \end{aligned}$$

Define a map $\varphi : V(F) \to [1, n-2]$ where

$$\begin{aligned} \varphi(v_i) &= \begin{cases} i, & \text{if } i \in [1, n-2], \\ 1, & \text{if } i = n-1. \end{cases} \\ \varphi(u) &= 1. \end{aligned}$$

The only pair which has a distance 3 is u and v_{n-2}. Since $u \sim v_{n-1} \sim v_2 \sim v_{n-2}$ is a rainbow path, then F is rainbow vertex-connected. Moreover, the only vertices with the same colors are v_1, v_{n-1}, and u. Observe that $d_F(v_1, v_{n-2}) = 1$, $d_F(v_{n-1}, v_{n-2}) = 2$, and $d_F(u, v_{n-2}) = 3$. Since $d_F(v_1, v_{n-2}) < d_F(v_{n-1}, v_{n-2}) < d_F(u, v_{n-2})$, then every vertex in F has a distinct rainbow codes. Therefore, $rvcl(F) = n - 2$.

Now, define a graph G^* which is given by

$$\begin{aligned} V(G^*) &= \{u_1, u_2\} \cup \{v_i \mid i \in [1, n-2]\}, \\ E(G^*) &= \{u_2 v_{n-2}\} \cup \{v_i v_j \mid i, j \in [1, n-2], i \neq j\}. \end{aligned}$$

We can see that G^* is a spanning subgraph of G_{10} and G_{12}. Precisely, G_{10} has $V(G_{10}) = V(G^*)$ and $E(G_{10}) = E(G^*) \cup \{u_1 u_2\}$. Meanwhile, G_{12} has $V(G_{12}) = V(G^*)$ and $E(G_{12}) = E(G^*) \cup \{u_1 v_{n-3}\}$. Let both of $id_{G_{10}} : V(G_{10}) \to V(G^*)$ and $id_{G_{12}} : V(G_{12}) \to V(G^*)$ be identity maps. Define a map $\varphi : V(G^*) \to [1, n-3]$ such that

$$\begin{aligned} \varphi(v_i) &= i, \qquad \text{for } i \in [1, n-2], \\ \varphi(u_i) &= i, \qquad \text{for } i \in \{1, 2\}. \end{aligned}$$

In this coloring, vertices with the same colors come in pairs that is (v_i, u_i) for $i \in \{1, 2\}$.

Now, consider a coloring $\varphi_{G_{10}} = id_{G_{10}} \circ \varphi$ on G_{10}. Any shortest path of length 3 must only have v_2 and v_{n-2} as the internal vertices. Since $\varphi_{G_{10}}(u_2) \neq \varphi_{G_{10}}(v_{n-2})$ then G_{10} is rainbow vertex-connected. Therefore, $rvcl(G_{10}) = n - 2$. Observe that

- $d_{G_{10}}(v_1, v_{n-4}) = 1 = d_{G_{10}}(v_2, v_{n-4})$,
- $d_{G_{10}}(u_1, v_{n-4}) = 3$,
- $d_{G_{10}}(u_2, v_{n-4}) = 2$.

Since we know that $d_{G_{10}}(v_1, v_{n-4}) < d_{G_{10}}(u_1, v_{n-4})$ and $d_{G_{10}}(v_2, v_{n-4}) < d_{G_{10}}(u_2, v_{n-4})$, then every rainbow codes of G_{10} is unique. This implies $rvcl(G_{10}) = n - 2$.

Lastly, consider a coloring $\varphi_{G_{12}} = id_{G_{12}} \circ \varphi$. The only pair of vertices that has a distance 3 in G_{12} is u_1 and u_2. Since $u_1 \sim v_{n-3} \sim v_{n-2} \sim u_2$ is a rainbow path, then G_{12} is rainbow vertex-connected. Moreover, for $i \in \{1, 2\}$ since

$$d_{G_{12}}(v_i, v_{n-4}) = 1 < 2 = d_{G_{12}}(u_i, v_{n-4}),$$

then each rainbow codes of G_{12} is unique. Therefore, $rvcl(G_{12}) = n - 2$.

Since we have shown that each graph G with $pd(G) = n-2$ has $rvcl(G) = n - 2$, then the theorem follows. □

Next, we will see some families of graph G having $rvcl(G) = |V(G)| - 3$. Let m_1, m_2, and m_3 be positive integers. Let $TS(m_1, m_2, m_3)$ be a graph with the vertex set

$$\begin{aligned} V(TS(m_1, m_2, m_3)) = \{u_i, v_j, x_k \mid i \in [1, m_1], j \in [1, m_2], \\ k \in [1, m_3]\} \end{aligned}$$

and the edge set

$$\begin{aligned} E(TS(m_1, m_2, m_3)) = \{u_i u_{i+1} \mid i \in [1, m_1 - 1]\} \\ \cup \{v_j v_{j+1} \mid j \in [1, m_2 - 1]\} \\ \cup \{x_k x_{k+1} \mid k \in [1, m_3 - 1]\} \\ \cup \{u_1 v_1, v_1 x_1, u_1 x_1\}. \end{aligned}$$

We can view the graph as a triangle whose vertices are identified with path of length m_i. We present an example of $TS(2, 4, 5)$ in Figure 3.

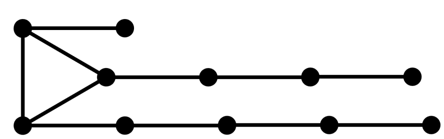

Fig. 3. The graph $TS(2, 4, 5)$ where the vertices of the triangle is identified with paths of length 2, 4, and 5 respectively.

Theorem 8. *For positive integers m_1, m_2 and m_3, let $TS(m_1, m_2, m_3)$ be a graph of order $n = m_1 + m_2 + m_3$. It holds that*

$$rvcl(TS(m_1, m_2, m_3)) = n - 3.$$

Proof. Let $G = TS(m_1, m_2, m_3)$. First, we will show the lower bound for $rvcl(G)$. For $y, z \in V(G)$, let $P(y, z)$ be a path with y and z as endpoints. Let $A = \{\varphi(v_i) \mid i \in [1, m_1 - 1]\}$, $B = \{\varphi(v_j) \mid j \in [1, m_2 - 1]\}$, and $C = \{\varphi(x_k) \mid k \in [1, m_3 - 1]\}$. We have $|A \cup B| = m_1 + m_2 - 2$ since they are the internal vertices of $P(u_{m_1}, v_{m_2})$. Similarly, $|B \cup C| = m_2 + m_3 - 2$ and $|A \cup C| = m_1 + m_3 - 2$ by considering $P(v_{m_2}, x_{m_3})$ and $P(u_{m_1}, x_{m_3})$. This implies $|A \cup B \cup C| = m_1 + m_2 + m_3 - 3 \leq rvcl(G)$.

To show $rvcl(G) \leq m_1 + m_2 + m_3 - 3$, consider a map $\varphi : V(G) \to [1, m_1 + m_2 + m_3 - 3]$ where

$$\varphi(u_i) = \begin{cases} i, & \text{if } i \in [1, m_1 - 1], \\ m_1 - 1, & \text{if } i = m_1 - 1, \end{cases}$$

$$\varphi(v_j) = \begin{cases} j + m_1 - 1, & \text{if } j \in [1, m_2 - 1], \\ m_1 + m_2 - 2, & \text{if } j = m_2 - 1, \end{cases}$$

$$\varphi(x_k) = \begin{cases} k + m_1 + m_2 - 1, & \text{if } k \in [1, m_3 - 1], \\ m_1 + m_2 + m_3 - 3, & \text{if } k = m_3 - 1. \end{cases}$$

Since every vertex in $V(G) \setminus \{u_{m_1}, v_{m_2}, x_{m_3}\}$ has a unique color, then G is rainbow vertex-connected. Moreover, we have

- $d_G(u_{m_1}, v_1) = m_1 > m_1 - 1 = d_G(u_{m_1 - 1}, v_1)$,
- $d_G(v_{m_2}, x_1) = m_2 > m_2 - 1 = d_G(v_{m_2 - 1}, x_1)$,
- $d_G(x_{m_3}, u_1) = m_3 > m_3 - 1 = d_G(x_{m_3 - 1}, u_1)$.

Therefore, all rainbow codes in G are pairwise distinct. This shows that $rvcl(TS(m_1, m_2, m_3)) = n - 3$. □

Now, let $V(K_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(P_4) = \{x_1, x_2, x_3, x_4\}$. For a vertex $x \in V(P_4)$, the graph $Amal(K_m, P_4; v_1 = x)$ is a graph obtained by

$$V(Amal(K_m, P_4; v_1 = x)) = V(K_m) \cup V(P_4),$$
$$E(Amal(K_m, P_4; v_1 = x)) = E(K_m) \cup E(P_4),$$

where $v_1 = x$. This is a complete graph of order m where one of the vertices is identified with a path of order 4. Using this notion, we can get another graph of order n whose locating rainbow vertex connection number is $n - 3$. There will be two distinct results based on the choice of x, when $x \in \{x_1, x_4\}$ and when $x \in \{x_2, x_3\}$. An illustration of $Amal(K_m, P_4; v_1 = x)$ is given in Figure 4.

Theorem 9. *Let $n \geq 7$ be an integer. Then*

$$rvcl(Amal(K_{n-3}, P_4; v_1 = x)) = n - 3$$

for any $x \in V(P_4)$.

Proof. Let $G = Amal(K_{n-3}, P_4; v_1 = x)$. Observe that $v_2, v_3, \ldots, v_{n-3}$ are twin vertices, hence they have distinct colors. If $\varphi(v_1) = \varphi(v_j)$ for some $j \in [2, n-3]$ and

$$\{\varphi(x_i) \mid i \in [1, 4]\} \subseteq \{\varphi(v_j) \mid j \in [2, n - 3]\},$$

then $c_\pi(v_1) = c_\pi(v_j)$. Therefore, either $\varphi(v_1) \neq \varphi(v_j)$ for every $j \in [2, n-3]$ or there exists $i \in [1, 4]$ such that $\varphi(x_i) \notin \{\varphi(v_j) \mid j \in [2, n - 3]\}$. In either case, locating rainbow vertex coloring of G must have at least $n - 3$ colors. Hence, $rvcl(G) \geq n - 3$.

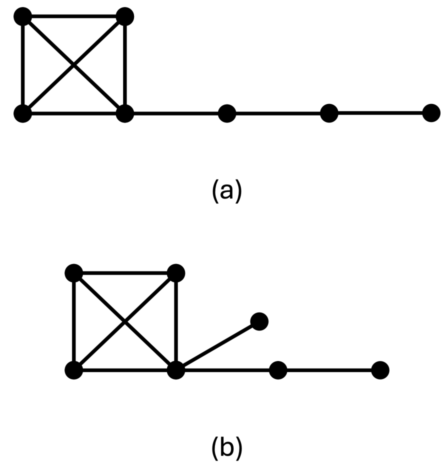

Fig. 4. The graph $Amal(K_m, P_4; v_1 = x)$ where (a) $x \in \{x_1, x_4\}$ and (b) $x \in \{x_2, x_3\}$.

To show the upper bound, consider the following cases.

Case 1. Let $x = x_1$. Define a map $\varphi : V(G) \to [1, n - 3]$ where

$$\varphi(v_j) = j, \qquad \text{for } j \in [1, n - 3],$$
$$\varphi(x_i) = i, \qquad \text{for } i \in [2, 4].$$

In any shortest path of G, only the vertices $A = \{v_1, x_2, x_3\}$ can be the internal vertices of such path. Since the colors of A are pairwise distinct, then G is rainbow connected. Further, two vertices with the same colors come only in pairs (v_i, x_i) for $i \in [2, 4]$. Observe that

- $d_G(v_2, \varphi^{-1}(4)) = 1 < 2 = d_G(x_2, \varphi^{-1}(4))$,
- $d_G(v_3, \varphi^{-1}(1)) = 1 < 2 = d_G(x_3, \varphi^{-1}(1))$,
- $d_G(v_4, \varphi^{-1}(1)) = 1 < 3 = d_G(x_4, \varphi^{-1}(1))$.

This shows that every rainbow codes of G are distinct.

Case 2. Let $x = x_2$. Let $\varphi : V(G) \to [1, n-3]$ be a map such that

$$\varphi(v_j) = j, \qquad \text{for } j \in [1, n - 3],$$

$$\varphi(x_i) = \begin{cases} 2, & \text{if } i = 1, \\ i, & \text{if } i \in \{3, 4\}. \end{cases}$$

Again, in any shortest path of G, only the vertices $B = \{v_1, x_3\}$ can be the internal vertices of such path. Since the colors of B are pairwise distinct, then G is rainbow vertex-connected. Moreover, two vertices with the same colors are (v_2, x_1), (v_3, x_3) and (v_4, x_4). Since we have

- $d_G(v_2, \varphi^{-1}(3)) = 1 < 2 = d_G(x_1, \varphi^{-1}(3))$,
- $d_G(v_3, \varphi^{-1}(2)) = 1 < 2 = d_G(x_3, \varphi^{-1}(2))$,
- $d_G(v_4, \varphi^{-1}(1)) = 1 < 2 = d_G(x_4, \varphi^{-1}(1))$,

then every rainbow codes of G are distinct.

Note that the graph obtained by $x = x_3$ is the same as in the case of $x = x_2$, since both x_2 and x_3 are vertices of degree two in P_4. Likewise, the graph obtained by $x = x_4$ is the same as in the case $x = x_1$, since both of them are leaves in P_4. Therefore, in any case we have $rvcl(G) = n - 3$. □

III. Concluding Remarks

In this paper, we investigated graphs whose locating rainbow vertex connection number is close to their order, specifically $rvcl(G) \in \{n, n-1, n-2, n-3\}$. By establishing connections to partition dimension, we identified several structural thresholds where distinguishability under chromatic constraints becomes most costly as in demanding nearly as many colors as vertices for some particular graphs.

Our results reveal that while $rvcl(G)$ coincides with $pd(G)$ for graphs of diameter at most two, the gap between the parameters may increase significantly when the diameter increases, as demonstrated by the existence of graphs with arbitrarily large gaps between their partition dimension and locating rainbow vertex connection number. The families $TS(m_1, m_2, m_3)$ and $Amal(K_{n-3}, P_4; v_1 = x)$ further illustrate how specific structural features force the parameter to $n-3$.

A natural direction for future work is a complete characterization of graphs with $rvcl(G) = n-1$. Further, determining the conditions of graphs with $rvcl(G) = n-2$ is also interesting.

References

[1] F. Anggalia, L. Yulianti, and D. Welyyanti, Batas atas rainbow connection number pada graf Buckminsterfullerene [The upper bound of rainbow connection number of Buckminsterfullerene graph], *J. Mat. UNAND*, **11**(1) (2022), 1–11. DOI: https://doi.org/10.25077/jmu.11.1.1-11.2022

[2] E. T. Baskoro and D. O. Haryeni, All graphs of order $n \geq 11$ and diameter 2 with partition dimension $n-3$, *Heliyon*, **6**(4) (2020), e03694.

[3] A. W. Bustan, A. N. M. Salman, and P. E. Putri, *On the locating rainbow connection number of a graph*, In J. Phys.: Conf. Ser. **1764** (2021), 012057.

[4] A. W. Bustan, A. N. M. Salman, P. E. Putri, and Z. Y. Awanis, On the locating rainbow connection number of trees and regular bipartite graphs, *Emerg. Sci. J.*, **7**(4) (2023), 1260–1273.

[5] A. W. Bustan, A. N. M. Salman, and P. E. Putri, Determining the locating rainbow connection numbers of vertex-transitive graphs, *Commun. Comb. Optim.*, in press. DOI: https://doi.org/10.22049/cco.2025.29742.2137

[6] G. Chartrand, E. Salehi, and P. Zhang, The partition dimension of a graph, *Aequationes Math.*, **59** (2000), 45–54.

[7] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang, Rainbow connection in graphs, *Math. Bohem.*, **133**(1) (2008), 85–98.

[8] C. Darayon and W. Tangjai, Rainbow vertex-connection number on a small-world Farey graph, *AKCE Int. J. Graphs Combin.*, **19**(1) (2022), 54–60. DOI: http://dx.doi.org/10.1080/09728600.2022.2057827

[9] D. Fitriani, A. N. M. Salman, and Z. Y. Awanis, Rainbow connection number of comb product of graphs, *Electron. J. Graph Theory* **10**(2) (2022), 461–474. DOI: https://dx.doi.org/10.5614/ejgta.2022.10.2.9

[10] F. Harary and R. Melter, On the metric dimension of a graph, *Ars Combin.*, **2** (1976), 191–195.

[11] D. O. Haryeni, E. T. Baskoro, and S. W. Saputro, Family of graphs with partition dimension three, *Indones. J. Combin.*, **8**(2) (2024), 64–75. DOI: http://dx.doi.org/10.19184/ijc.2024.8.2.1

[12] D. O. Haryeni, M. Ridwan, and E. T. Baskoro, Graphs of order n with partition dimension $n-3$, *IAENG Int. J. Appl. Math.*, **53** (2023), no. 1, 152–161.

[13] Haspika, Hasmawati, and N. Aris, The partition dimension on the grid graph, *J. Mat. Stat. Komp.*, **19** (2023), no. 2, 351–358. https://doi.org/10.20956/j.v19i2.23904

[14] M. Imrona, A. N. M. Salman, S. Uttunggadewa, and P. E. Putri, On the locating rainbow connection number of the comb product with complete graphs or trees, In *Combin. Graph Theory Comput.*, **448** (2024), 203–214.

[15] M. Krivelevich and R. Yuster, The rainbow connection of a graph is (at most) reciprocal to its minimum degree, *J. Graph Theory*, **63**(3) (2009), 195–191.

[16] D. Kuziak and I. G. Yero, Metric dimension related parameters in graphs: A survey on combinatorial, computational and applied results, *arXiv:2107.04877v1*, 2021.

[17] T. K. Maryati, D. Sobiruddin, and F. F. Hadiputra, On metric dimension of edge comb product of symmetric graphs, *J. Mat. UNAND*, **13**(4) (2024), 349–357.

[18] T. K. Maryati, D. Sobiruddin, M. Fatra, and F. F. Hadiputra, On metric dimension of edge comb product of vertex-transitive graphs, *Trans. Combin.*, **14**(1) (2025), 45–64.

[19] K. H. Rosen, *Discrete Mathematics and its Applications*, McGraw-Hill, NY, 2012.

[20] P. Slater, Leaves of trees, *Proc. 6th Southeastern Conf. on Comb. Graph Theory Comput.*, **14** (1975), 549–559.

[21] R. C. Tillquist, R. M. Frongillo, and M. E. Lladser, Getting the lay of the land in discrete space: A survey of metric dimension and its applications, *SIAM Rev.*, **65**(4) (2023), 919–962.

[22] I. Tomescu, Discrepancies between metric dimension and partition dimension of a connected graph, *Discrete Math.*, **308** (2008), 5026–5031.

[23] T. Vetrík, M. Imran, M. Knor, and R. Škrekovski, The metric dimension of the circulant graph with $2k$ generators can be less than k, *Journal of King Saud University - Science*, **35**(7) (2023), 102834. DOI: https://doi.org/10.1016/j.jksus.2023.102834

[24] D. Welyyanti, A. Arsyad, and L. Yulianti, Dimensi metrik amalgamasi graf theta [Metric dimension of amalgamation of theta graphs], *Limits: J. Math. Appl.*, **20**(2) (2023), 241–253. DOI: http://dx.doi.org/10.12962/limits.v20i2.16359

[25] N. I. Yahya, A. Fatmawati, Nurwan, and S. K. Nasib, Rainbow vertex-connection number on comb product operation of cycle graph C_4 and complete bipartite graph K_{3n}, *Barekeng: J. Math. Appl.*, **17**(2) (2023), 673–684. DOI: https://doi.org/10.30598/barekengvol17iss2pp0673-0684