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Diffusion-Advection Reaction in Gierer-Meindhart System

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Abstract

Gierer-Meinhardt system is a mathematical model commonly used to describe chemical and biological phenomena, specifically the interaction between two types of molecules: activator and inhibitor. Diffusion reactions are processes where chemical molecules move from areas of high concentration to areas of low concentration randomly. In this research, the Gierer Meinhardt system with advection components is considered. The mathematical model developed consists of partial differential equations (PDE) for two variables, namely the activator and the inhibitor, with the addition of an advection component that depends on the flow speed. Studies show that the presence of advection significantly affects the characteristics of the patterns formed. Advection reactions cause more regular and oriented patterns compared to patterns produced only by diffusion. This research uses theoretical analysis methods and literature studies to analyze the impact of advection-diffusion reactions on the Gierer Meinhardt system. In this research, it is concluded that the equilibrium point $E_1 = \left(\frac{\gamma^2+c_0}{\mu}, \frac{\gamma^2+c_0}{\mu\gamma}\right)$ which was initially stable when $c_0 > \frac{\mu^2}{4}$ becomes stable only when $n^3(n^2d - 2\delta_1 d\varepsilon) > -\frac{2\gamma^3}{\varepsilon}\left(\frac{\delta_1}{\delta_2}\right)$ and $\delta_1 < \frac{n^2+n^3d}{2\varepsilon}$ if $\varepsilon = \frac{\gamma^2+c_0}{\mu}$.

***Keywords**: Gierer-Meindhart system, activator and inhibitor, reaction-diffusion, reaction-advection, stability analysis*

1 Introduction

Mathematical models of partial differential equations are often used to describe the dynamics of biological pattern formation [1]. The Gierer-Meinhardt system is one of the mathematical models used to describe patterns in biology, particularly in organism development. This model was developed by Alfred Gierer and Hans Meinhardt in 1972 [2]. Gierer and Meinhardt introduced a prototype model of coupled reaction-diffusion equations, which describe the interaction between two substances, namely an activator and an inhibitor. The equilibrium point in the Gierer-Meinhardt system is the condition where the concentrations of the activator and inhibitor do not change over time. The stability of this system can be analyzed using both linear and nonlinear analysis. Nonlinear analysis of the Gierer-Meinhardt system is carried out by considering the system as a partial differential system. In this analysis, the system is considered stable if it has a stable equilibrium point [3].

Many studies have investigated the diffusion process under various conditions in the Gierer-Meinhardt system, see [3], [4], [5], [6], [7], [8]. Most of these studies only discuss the diffusion process under different conditions. More complex analysis is then found in [9], [10], [11], which compares diffusion-advection reactions (DAR) from various models. The advection-diffusion equation is a mathematical equation designed to study the transport phenomenon of pollutants [12]. This equation combines the properties of the advection equation and the diffusion equation. The advection equation is a type of partial differential equation that models the movement of a concentration in a flowing fluid, assuming that the concentration does not undergo a diffusion process within the fluid [7]. The addition of an advection term can alter the patterns formed in the system. The patterns formed can be spiral patterns, grid patterns, or other types of patterns. The advection term can also increase or decrease the stability of the system.

This research will examine a more complex Gierer-Meinhardt system by incorporating an additional component in the form of an advection term into the system. This approach aims to understand the dynamics of more intricate pattern formation and the influence of advection factors on the distribution and interaction between the activator and inhibitor in the modified reaction-diffusion model. The Gierer-Meindhart system with diffusion reaction is given as

$$\begin{cases} u_t = D_u u_{xx} + \rho_0 \rho + \frac{c\rho u^2}{v^2} - \mu u, \\ v_t = D_v v_{xx} + c'\rho' u - \gamma v. \end{cases} \tag{1}$$

For simplicity by the substitution of u, v, μ, t, γ for $u^*, v^*, \mu^*, t^*, \gamma^*$, we rewrite system (1) into

$$\begin{cases} u_t = u_{xx} + c_0 + \frac{u^2}{v^2} - \mu u, \\ v_t = d v_{xx} + u - \gamma v, \end{cases} \tag{2}$$

by using the non-dimensional scaling trasnformation:

$$u = \frac{c\rho D_u}{(c'\rho')^2} u^*, \quad v = \frac{c\rho}{c'\rho'} v^*, \quad c_0 = \frac{\rho_0 (c'\rho')^2}{c(D_u)^2},$$

$$t = \frac{t^*}{D_u}, \quad \mu = D_u \mu^*, \quad \gamma = D_u \gamma^*, \quad D_v = d D_u,$$

and subject to the initial conditions

$$u(x,0) = u_0(x), \;\; v(x,0) = v_0(x),$$

where $c_0, \mu, \gamma, d > 0, u, v \geq 0, x \in (0, \pi), t \in (0, +\infty)$. In the sequel, we will study system (2) subjected to the Neumann boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0, \qquad v_x(0,t) = v_x(\pi,t) = 0. \tag{3}$$

that can represent a situation where the population cannot freely exchange individuals with the external environment.

Then, advection terms (a) will be added to the system to study their effects on the stability of the system. So, the formulated system is

$$\begin{cases} u_t = u_{xx} + au_x + c_0 + \frac{u^2}{v^2} - \mu u, & x \in (0,\pi), t \in (0,+\infty), \\ v_t = dv_{xx} + av_x + u - \gamma v, & x \in (0,\pi), t \in (0,+\infty), \\ u_x(0,t) = u_x(\pi,t) = 0, & t > 0, \\ v_x(0,t) = v_x(\pi,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in (0,\pi), \\ v(x,0) = v_0(x), & x \in (0,\pi). \end{cases} \tag{4}$$

This article contributes to understanding the dynamics of patterns in the Gierer-Meinhardt system modified by the addition of an advection component. Unlike previous studies that only examined patterns in diffusion-based systems, this research investigates the combined effects of diffusion-advection reactions on the system's stability and pattern formation. The novelty of this study lies in the stability analysis under Neumann boundary conditions and the exploration of the influence of advection velocity on the characteristics of the patterns formed.

2 Research Methods

This research employs a literature review method to examine previous studies related to the Gierer-Meinhardt system and the stability analysis of partial differential equation systems. Specifically, the review focuses on understanding the mathematical modeling and stability conditions of reaction-diffusion-advection systems, including their equilibrium points, bifurcation behaviors, and pattern formation dynamics. The research design encompasses a comprehensive review of pertinent literature, data collection from credible sources, and systematic data analysis to achieve the research objectives. This study involves the modeling of the Gierer-Meinhardt system, incorporating diffusion-advection reactions. The mathematical model of this system is formulated based on partial differential equations that describe the interaction between the activator and the inhibitor within the system. Data sources for this research include journal articles, books, and other scientific publications that discuss the Gierer-Meinhardt system and stability analysis. The steps undertaken in this study are illustrated in the following flowchart.

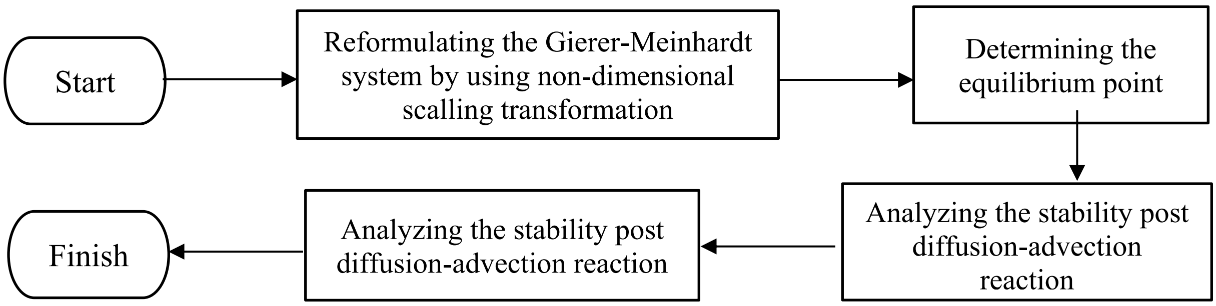

Figure 1. Flowchart of the Analysis Steps in the Gierer-Meinhardt System with Diffusion-Advection Reactions

3 Results and Discussion

We begin by considering system (2) in the absence of the diffusion and avection terms, namely:

$$\begin{cases} u_t = c_0 + \frac{u^2}{v^2} - \mu u := f(u,v), \\ v_t = u - \gamma v := g(u,v). \end{cases} \quad (5)$$

First we will discuss the existence and uniqueness of the equilibrium of system (5). The corresponding results are presented in the following theorem.

Theorem 1. *The ODE system* (5) *has a unique positive equilibirum* (u_s, v_s), *where* $u_s = \frac{\gamma^2+c_0}{\mu}, h_s = \frac{u_s}{\gamma}$.

Proof. The equilibrium (u_s, v_s) satisfies the following equations:

$$\begin{cases} f(u,v) = 0, \\ g(u,v) = 0. \end{cases} \quad (6)$$

Define $\varphi(u) = c_0 + \frac{\mu^2\gamma^2}{(c_0+\gamma^2)^2}u^2 - \mu u, u \in [0, +\infty)$, then $\varphi(u) = 0$. This function is a quadratic function of u. Since the coefficient of u^2 is positive $(\mu^2\gamma^2)$, this parabola opens upwards. For each c_0, there is exactly one positive solution u that satisfies this equation, which means $\varphi(u)$ has one zero point in $[0, +\infty)$. Then existence of the equilibirum point will be determined by substituting $u = \gamma v$ from $v_t = 0$ to $u_t = 0$. We get $u_s = \frac{\gamma^2+c_0}{\mu}$ and $h_s = \frac{u_s}{\gamma}$.

System Stability without Diffuison-Advection Reaction

In the research, we mainly consider the stability of the only positive equilibrium (u_s, v_s) for system (5). The linearized system of (5) at the positive equilibrium $E_1(u_s, v_s)$ is

$$J(u_s, v_s) = \begin{pmatrix} \frac{2\mu\gamma^2}{\gamma^2+c_0} - \mu & \frac{-2\mu\gamma^3}{\gamma^2+c_0} \\ 1 & -\gamma \end{pmatrix}. \quad (7)$$

So, the corresponding characteristic equation is

$$\lambda = \frac{tr(J) \pm \sqrt{tr\,(J)^2 - 4\det(J)}}{2},$$

where

$$\det(J) = \mu\gamma > 0, \;\; tr(J) = \frac{2\mu\gamma^2}{\gamma^2+c_0} - \mu - \gamma < 0. \tag{8}$$

Theorem 2. *The unique positive equilibrium* (u_s, v_s) *of system* (5) *is stable if* $c_0 > \frac{\mu^2}{4}$.

Proof. For stability, the real parts of both eigenvalues λ must be negative. The stability condition can be determined by checking the signs of $tr(J)$ and det (J). We need $tr(J) < 0$ and $\det(J) > 0$. Given $\det(J) = \mu\gamma,$ because μ and γ is positive parameters, thus $\det(J)$ will be greater than 0. Given $tr(J) = \frac{2\mu\gamma^2}{\gamma^2+c_0} - \mu - \gamma < 0$. This condition will be satisfied if $c_0 > \frac{\mu^2}{4}$. Hence, the unique positive equilibrium (u_s, v_s) of system (3.1) is stable if $c_0 > \frac{\mu^2}{4}$.

System Stability with Diffusion-Advection Reaction

To analyze the stability with diffusion-advection reaction system, we use transformation $u(x,t) = w(x,t)e^{\alpha_1 x - \beta_1 t}$ and $v(x,t) = z(x,t)e^{\alpha_2 x - \beta_2 t}$ into (4). So, that it is obtained

$$\begin{aligned} w_t &= w_{xx} + (2\alpha_1 + a)w_x + (\alpha_1^2 + a\alpha_1 - \mu + \beta_1)w + c_0 e^{-\alpha_1 x + \beta_1 t} + \frac{w^2}{z^2} e^{2(\alpha_1-\alpha_2)x - 2(\beta_1-\beta_2)t}, \\ z_t &= dz_{xx} + (2d\alpha_2 + a)z_x + (d\alpha_2^2 + a\alpha_2 - \gamma + \beta_2)z + we^{(\alpha_1-\alpha_2)x-(\beta_1-\beta_2)t}. \end{aligned} \tag{9}$$

Transformation is carried out $\alpha_1 = \frac{-a}{2},\; \alpha_2 = \frac{-a}{2d},\; \beta_1 = \frac{a^2}{4} + \mu,\; \beta_2 = \frac{a^2}{4d} + \gamma$ into (9), provide

$$\begin{aligned} w_t &= w_{xx} + c_0 e^{\frac{a}{2}x + \left(\frac{a^2}{4}+\mu\right)t} + \frac{w^2}{z^2} e^{\left(\frac{-ad+a}{4}\right)x - \left(\frac{a^2 d - a^2 + \mu - \gamma}{2d}\right)t} \\ z_t &= dz_{xx} + we^{\left(\frac{-ad+a}{2d}\right)x - \left(\frac{a^2 d - a^2}{4d} + \mu - \gamma\right)t} \end{aligned} \tag{10}$$

we consider a small pertubation of the equilibrium,

$$w(x,t) = w_s + \epsilon\widetilde{w}(x,t), \qquad z(x,t) = z_s + \epsilon\tilde{z}(x,t), \tag{11}$$

where ϵ are the small. Linearizing the function (11) at (u_s, v_s) using a Taylor series gives

$$\begin{cases} w_t = w_{xx} + \frac{2w_s}{z_s^2} w - \frac{2w_s^2}{z_s^3} z, \\ z_t = dz_{xx} + we^{\left(\frac{-ad+a}{2d}\right)x - \left(\frac{a^2 d + a^2 - \mu + \gamma}{4d}\right)t}, \\ w_x(0,t) = w_x(\pi,t) = 0, w(x,0) = w_0(x) - w_s, x \in (0,\pi), t > 0, \\ z_x(0,t) = z_x(\pi,t) = 0, z(x,0) = z_0(x) - z_s, x \in (0,\pi), t > 0. \end{cases} \tag{12}$$

Since the perturbation functions must satisfy the linear diffusion equation together with the related initial values, these perturbations are decomposed into a series of sin and cos functions with different spatial frequencies. Furthermore, because system (12) has Neumann boundary conditions, Fourier transformation will be used with pertubations as follows

$$w(x,t) = \sum_{n\in\mathbb{Z}} \xi_n(t)\cos nx\,,\ \ z(x,t) = \sum_{n\in\mathbb{Z}} \eta_n(t)\cos nx\,. \tag{13}$$

By substituting (12) into the linearized system (10), canceling out a common factor of $\cos nx$, and transform back $\alpha_1, \alpha_2, \beta_1$, and β_2 for each n leads to the system

$$\begin{pmatrix} \dot{\xi}_n(t) \\ \dot{\eta}_n(t) \end{pmatrix} = J_n \begin{pmatrix} \xi_n(t) \\ \eta_n(t) \end{pmatrix}, \tag{14}$$

where

$$J_n = \begin{pmatrix} -n^2 + \frac{2w_s}{z_s^2} & \frac{-2w_s^2}{z_s^3} \\ e^{(\alpha_1-\alpha_2)x-(\beta_1-\beta_2)t} & -n^3 d \end{pmatrix}. \tag{15}$$

Reverting transformation $u(x,t) = w(x,t)e^{\alpha_1 x-\beta_1 t}$ and $v(x,t) = z(x,t)e^{\alpha_2 x-\beta_2 t}$ with $w(x,t) = u(x,t)e^{-\alpha_1 x+\beta_1 t}$ and $z(x,t) = v(x,t)^{-\alpha_2 x+\beta_2 t}$ and gives

$$J_n = \begin{pmatrix} -n^2 + 2u_s e^{-\alpha_1 x+\,\beta_1 t} & \frac{-2u_s^2 e^{2(-\alpha_1 x+\beta_1 t)}}{v^3 e^{3(-\alpha_2 x+\beta_2 t)}} \\ e^{(\alpha_1-\alpha_2)x-(\beta_1-\beta_2)t} & -n^3 d \end{pmatrix}. \tag{16}$$

The stability of system (16) can be studied from the eigenvalues λ_n of the matrix J_n. By substituting $E_1\left(\frac{\gamma^2+c_0}{\mu}, \frac{\gamma^2+c_0}{\mu\gamma}\right)$ in (12), we obtain

$$J_n = \begin{pmatrix} -n^2 + 2\left(\frac{\gamma^2+c_0}{\mu}\right)e^{-\alpha_1 x+\,\beta_1 t} & \frac{-2\mu\gamma^3}{\gamma^2+c_0}\left(e^{-2\alpha_1 x+2\beta_1 t+3\alpha_2 x-3\beta_2 t}\right) \\ e^{(\alpha_1-\alpha_2)x-(\beta_1-\beta_2)t} & -n^3 d \end{pmatrix}. \tag{17}$$

If it is assumed $\delta_1 = e^{-\alpha_1 x+\,\beta_1 t}$, $\delta_2 = e^{-\alpha_2 x+\,\beta_2 t}$, and $\varepsilon = \frac{\gamma^2+c_0}{\mu}$ provide

$$J_n = \begin{pmatrix} -n^2 + 2\delta_1\varepsilon & -\frac{2\gamma^3}{\varepsilon}\left(\frac{\delta_1^2}{\delta_2^3}\right) \\ \frac{\delta_2}{\delta_1} & -n^3 d \end{pmatrix} \tag{18}$$

Theorem 3. *If d is a constant diffusion term, a_1 and a_2 are constant advection term, and the system* (9) *reaches an equilibrium point at $E_1\left(\frac{\gamma^2+c_0}{\mu}, \frac{\gamma^2+c_0}{\mu\gamma}\right) = E_1\left(\varepsilon, \frac{\varepsilon}{\gamma}\right)$, then the system will be stable if*

i. $n^3(n^2 d - 2\delta_1 d\varepsilon) > -\frac{2\gamma^3}{\varepsilon}\left(\frac{\delta_1}{\delta_2}\right)$

ii. $\delta_1 < \frac{n^2+n^3 d}{2\varepsilon}$

Proof. For stability, the real parts of both eigenvalues λ must be negative. The stability condition can be determined by checking the signs of $tr(J)$ and $\det(J)$. We need $tr(J) < 0$ and $\det(J) > 0$. Find $\det(J_n)$ and $tr(J_n)$ from matrix (18), then gives

$$\det(J_n) = n^5 d - 2\delta_1 n^3 d\,\varepsilon + \frac{2\gamma^3}{\varepsilon}\left(\frac{\delta_1}{\delta_2^2}\right), \tag{19}$$

$$tr(J_n) = -n^2 + 2\delta_1\varepsilon - n^3 d. \tag{20}$$

Because $d, \delta_1, \delta_2, \epsilon, \gamma$ are positive parameters, then $n^3(n^2 d - 2\delta_1 d\varepsilon) > -\frac{2\gamma^3}{\varepsilon}\left(\frac{\delta_1}{\delta_2}\right)$ satisfies the condition $\det(J_n) > 0$ and $\delta_1 < \frac{n^2+n^3 d}{2\varepsilon}$ satisfies the condition $tr(J_n) < 0$.

4 Conclusion

In this research, it was conclude that the diffusion-advection reaction can affect the stability of equilibrium point $E_1\left(\frac{\gamma^2+c_0}{\mu}, \frac{\gamma^2+c_0}{\mu\gamma}\right)$ in the Gierer-Meindhart system. The equilibrium point which was initially stable when when $c_0 > \frac{\mu^2}{4}$ becomes stable only when $n^3(n^2 d - 2\delta_1 d\varepsilon) > -\frac{2\gamma^3}{\varepsilon}\left(\frac{\delta_1}{\delta_2}\right)$ and $\delta_1 < \frac{n^2+n^3 d}{2\varepsilon}$ if $\varepsilon = \frac{\gamma^2+c_0}{\mu}$. Advection components can have a significant impact on the stability and patterns of the Gierer-Meinhardt system. Therefore, further research is recommended to develop more efficient and accurate numerical methods to simulate the patterns formed in the system with more complex parameters, further investigate the influence of external factors, such as variations in environmental conditions or fluctuations in system parameters (e.g., advection speed or diffusion rate), on the patterns formed and the stability of the system, conduct further studies on bifurcation and pattern transitions in this modified system.

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